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Optimal strategies for an economic-agent in a continuous time model within social security system

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Abstract

In this Thesis we introduce a continuous life-time model for an economic-agent whose lifetime is uncertain and who invests his total savings in a financial market composed of one risk-free asset and one risky asset while buying life-insurance to protect his family in case of premature death. We also assume that this economic-agent contributes in the social security system, which can be private or governmental sector. This economic-agent is then faced with the problem of finding strategies that maximize the expected utility obtained from consumption, the size of his estate in the event of premature death and the size of his fortune at time of retirement if he lives that long. We use dynamic programming techniques to derive a second order nonlinear partial differential equation. Using CRRA utility functions together with the effect of the social security system, we characterize the optimal strategies of consumption, investment and life insurance selection.

الملخص

في هذه الرسالة نقدم نموذج حياة متصل لوكيل اقتصادي تتمثل حياته في متغير عشوائي متصل حيث يستثمر جميع ثروته في سوق مالي يتكون من أصول أحدها محفوف بالمخاطر والآخر غير محفوف بالمخاطر بحيث يرغب هذا الوكيل بشراء تأمين حياة لحماية عائلته من المخاطر المستقبليه بالاضافه الى مشاركته في نظام الضمان الإجتماعي. الهدف الذي تسعى اليه هذه الرساله هو ايجاد الحل النموذجي للإستهلاك والإستثمار والتأمين على الحياة للوكيل في ظل اشتراكه في نظام الضمان الاجتماعي من أجل تحقيق منفعة اقتصادية متوقعة و ذلك باستخدام طرق ديناميكيه مبرمجة.

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Symbols

$\mathbb{R}$	Real numbers
$\mathbb{R}^n$	Space of n dimensions
$\mathbb{R}^+$	Positive real numbers
$\mathbb{R}_0^+$	Non-negative real numbers
$\mathcal{F}$	Sigma algebra on Ω
$\mathcal{F}_t$	Filtration
ϕ	Empty set
$\subseteq$	Subset
$\inf$	Infimum
$\sup$	Supermum
$\min$	Minimum
$\max$	Maximum
$\otimes$	Tensor product
f^{-1}	Preimage
ODE	Ordinary differential equation
SDE	Stochastic differential equation
PDE	Partial differential equation
$\mathbb{P}$	Probability measure
$argmin$	Argument minimum
$L.H.S$	Left hand side
BM	Brownian motion
DPP	Dynamic programming principle
$r.v$	Random variable
$CRRA$	Constant relative risk aversion
$\mathbb{E}$	Expectation

Chapter 1

Introduction

1.1 Literature Review

In this Thesis we handle a stochastic optimal control problem with the help of dynamic programming technique this technique is a recursive relation used to derive a nonlinear partial differential equation of order two, known as Hamilton-Jacobi-Bellman equation(HJB), whose solution is the value function we are looking for. For further readings we refer the reader to [11, 29, 30].

The history of dynamic programming principle backs to the great work of Bellman in 1950s [19–21]. Later on, it was developed and extended to conclude stochastic control problems like those papers in [4, 8, 9, 22]. The portfolio optimization problem is of critical importance in both theory and practice. For one thing, it is a stochastic control problem that can be solved via different approaches like dynamic programming principle. For another, the solution to the problem is a major concern for both individual and institutional investors, who need to allocate the wealth among various asset classes over a certain/ uncertain time horizon. Since the 1960s, there has been an intensive research in this area, and the problem of finding the optimal strategies for an economic-agent with uncertain lifetime has become of an interest for many scholars. For instance, Yarri [14] dealt with optimal consumption investment with uncertain lifetime. His work was extended by Hakansson [16, 17] to consider a discrete time including risky assets. Merton in [23, 24] studied the optimal consumption, investment without life insurance. Also Richard [27] combined the earlier work to obtain a continuous-time model for optimal consumption, investment and life insurance selection and purchase. Pliska and Ye [26] studied the optimal investment-consumption-insurance problem for an wage earner with an unbounded random lifetime. Huang et al [5] considered the problem with stochastic income which was correlated with risky asset. Duarte et al [7] generalised the complete market with a single risk asset to the incomplete market with multiple risky assets for the problem. Several studies considered a stochastic optimal control problem with random time horizon. For instance, the work done by Duarte et al [2]. In that work, an economic-agent with a random lifetime needs to find the optimal investment and life insurance from a single insurance company. In [31] Shenab and Weib studied an optimal investment, consumption and life insurance with random unbounded parameters. Mousa et al [13] studied the problem of finding optimal strategies for a wage-earner whose lifetime is uncertain. We will analyse this paper [13] with more details in this Thesis.

1.2 Basic Preliminaries

In this section, we introduce some basic concepts of stochastic differential equations.

Definition 1. *[6][* ***Random Experiment*** *]*
An experiment is called random experiment if its outcomes cannot be predicted.

As an example of random experiments include a roll of a die and toss of a coin.

Definition 2. *[6][* ***Sample Space*** *]*
The set of all possible outcomes of a random experiment is called the sample space denoted by Ω.

Definition 3. *[6] Any subset of the sample space* Ω *is called an event and any element that belongs to* Ω *is called sample point.*

Definition 4. *[3][* $\sigma-$ ***Algebra*** *]*
A $\sigma-algebra$ *is a collection* $\mathcal{F}$ *of subsets of* Ω *with these properties:*

- $\emptyset, \Omega \in \mathcal{F}$.
- *If* $A \in \mathcal{F}$*, then* $A^c \in \mathcal{F}$.
- *If* $A_1, A_2, ... \in \mathcal{F}$*, then*

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{F}.$$

That is, $\mathcal{F}$ is closed under complement and countable union.

Definition 5. *[3][* ***Measurable Space*** *]*
Let Ω *be a given set and* $\mathcal{F}$ *is a* σ*- algebra of subsets of* Ω*, then a pair* $(\Omega, \mathcal{F})$ *is called a measurable space.*

Example 1. *Let* $\Omega = \{a, b, c, d\}$ *be a discrete set. Consider the* σ *-algebra given by*

$$\mathcal{F} = \{\emptyset, \Omega, \{a, b\}, \{c, d\}\}.$$

Hence, $(\Omega, \mathcal{F})$ *is measurable space.*

Example 2. *Let* $\omega = (0, 1]$ *and* A *is a collection of subsets of* ω *which are the finite unions of disjoint intervals of the form (a, b] plus the null set. Then,* A *is not* $\sigma-$ *algebra since for example, if we take* $A_n = (0, \frac{n}{n+1}]$, $n = 1, 2, ...$ *then,* $A_n \in A$ *but* $\bigcup_{n=1}^{\infty} = (0, 1) \notin A$.

Definition 6. *[3][* ***Probability Measure****]*
Let $\mathcal{F}$ be a σ -algebra of subsets of Ω. We call $\mathbb{P}:\mathcal{F}\rightarrow[0,1]$ a probability measure on the measurable space $(\Omega,\mathcal{F})$ provided:

- $\mathbb{P}(\emptyset)=0, \quad \mathbb{P}(\Omega)=1.$
- $0\leq\mathbb{P}(A)\leq 1,\ \forall A\in\mathcal{F}.$
- *If A_1, A_2 ,... are pair-wise disjoint sets in $\mathcal{F}$ (i.e.$A_i\bigcap A_j=\emptyset, \quad i\neq j$), then*

$$\mathbb{P}(\bigcup_{k=1}^{\infty}A_k)=\sum_{k=1}^{\infty}\mathbb{P}(A_k).$$

Definition 7. *[3][* ***Probability Space****]*
Let Ω be any set, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and $\mathbb{P}$ is a probability measure. Then a triple $(\Omega,\mathcal{F},\mathbb{P})$ is called a probability space.

Definition 8. *[18][* ***Complete Probability Space****]*
A probability space $(\Omega,\mathcal{F},\mathbb{P})$ is called a complete probability space if $\forall\ A\in\mathcal{F}$ with $\mathbb{P}(A)=0,\ \forall\ G\subseteq A$ then $G\in\mathcal{F}$.

Remark 1. *[3]*

1. *Given a subset A of Ω. If $A\in\mathcal{F}$, then A is called $\mathcal{F}-$measurable set.*
2. *$\mathbb{P}(A)$ is the probability of the event A.*

Definition 9. *[18][* ***Generated*** *σ-* ***Algebra****]*
Given any family $\mathcal{A}$ of subsets of Ω, then there is a smallest σ- algebra $\mathcal{H}_{\mathcal{A}}$ containing $\mathcal{A}$, namely

$$\mathcal{H}_{\mathcal{A}}=\bigcap\{\mathcal{H},\mathcal{H}\ is\ \sigma-algebra\ of\ \Omega,\ \mathcal{A}\subseteq\mathcal{H}\}.$$

Note that $\mathcal{H}_{\mathcal{A}}$ is also called the σ- algebra generated by $\mathcal{A}$.

Definition 10. *[3][* ***Borel*** *$\sigma-$* ***Algebra****]*
The Borel σ -algebra $\mathcal{B}$ is the smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all open sets. We remark that the elements $B\in\mathcal{B}$ are called Borel measurable sets or simply Borel sets.

Example 3. *Let $\Omega=(0,1]$ and $b\in\ \Omega$, then the singleton $\{b\}$ is Borel.*

$$\{b\}=\bigcap_{n=1}^{\infty}[(b-\frac{1}{n},b+\frac{1}{n})\bigcap\Omega],$$

and using the fact that any countable intersection of Borel set is Borel so this leads to the fact that $(a,b]$, $[a,b]$, $[a,b)$ are also Borel.

Definition 11. *[3][**Random variable**]*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space. A mapping*

$$\mathbf{X} : \Omega \to \mathbb{R}^n$$

is called n-dimensional random variable. If for each $B \in \mathcal{B}$, *we have*

$$\mathbf{X}^{-1}(B) = \{w \in \Omega, X(w) \in B\} \in \mathcal{F},$$

then we say that $\mathbf{X}$ *is* $\mathcal{F}$ *-measurable*

Definition 12. *[3][**Indicator function**]*
Let $A \in \mathcal{F}$. *Then the function*

$$\mathbf{1_A}(w) := \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A \end{cases}$$

is called the indicator function of A.

Definition 13. *[3] Let* $\mathbf{X} : \Omega \to \mathbb{R}^n$ *be a random variable. Then*

$$\mathcal{F}(\mathbf{X}) := \{\mathbf{X}^{-1}(B) | B \in \mathcal{B}\}$$

is called the σ*-algebra generated by the random variable* $\mathbf{X}$.

The next definition is talking about random variables depending upon time.

Definition 14. *[18] [**Stochastic process**]*

1. *A collection of random variables* $\{\mathbf{X}(t) | t \in T\}$ *defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and takes on values on a measurable space* $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ *is called a stochastic process. We may refer to the stochastic process by* $X(\cdot)$ *representing the randomness in the variable* X.

2. *For each point* $\omega \in \Omega$, *the mapping* $t \to \mathbf{X}(t, \omega)$ *is the corresponding sample path.*

The interval of T is usually the halfline $[0, \infty)$, but it may be an interval $[a, b]$, the non-negative integers and even subsets of $\mathbb{R}^n$ for $n \geq 1$.

Now suppose we run an experiment for a given state $\omega_1 \in \Omega$ then the random values of $\mathbf{X}(t, \omega_1)$ as time evolves, form a sample path $\mathbf{X}(t, \omega_1)$ for $t \geq 0$. If we repeat the experiment with a different state $\omega_2 \in \Omega$, we will in general observe a different sample path $\mathbf{X}(t, w_2)$.

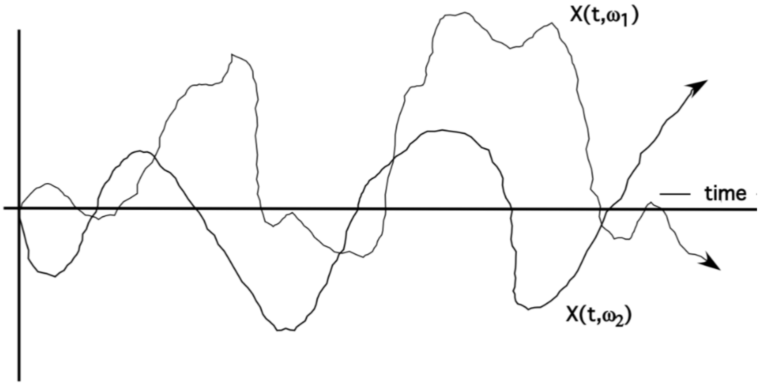

Figure 1.1: Two Sample Paths of The Stochastic Process $X(t)$

In our framework, we are interested only in the continuous random variables and one of the main topics related that is the cumulative distribution function defined below.

Definition 15. [3][***Cumulative Distribution Function***]
The cumulative distributionfunction of $\mathbf{X}$ *denoted by CDF is the function* $F_{\mathbf{X}} : \mathbb{R} \to [0,1]$ *defined by* $F_X(x) := P(X \leq x)$ *for all* $x \in \mathbb{R}$.

Most of the information about random experiment is determined by the behaviour of $F_X(x)$.
Below we illustrate some properties of the distribution function $F_X(x)$.

1. $0 \leq F_X(x) \leq 1$ for every $x \in \mathbb{R}$.
2. If $x_1 < x_2$ then $F_X(x_1) < F_X(x_2)$, i.e it is a nondecreasing function.
3. $\lim\limits_{x \to \infty} F_X(x) = F(\infty) = 1$.
4. $\lim\limits_{x \to -\infty} F_X(x) = F(-\infty) = 0$.
5. $\lim\limits_{x \to a^+} F_X(x) = F_X(a^+) = F_X(a)$ i.e $F_X(x)$ is right continuous.

The CDF of a continuous r.v X can be written as

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy,$$

where $f(x)$ is the corresponding probability density function denoted by pdf. Moreover,

$$P(a \leq X \leq b) = \int_a^b f(y)dy = F_X(b) - F_X(a).$$

One of the most fundamental concepts of probability theory and mathematical statistics is the expectation of a r.v which represents the center value of random variables.

Definition 16. [3] *If X is a r.v defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *then the expected value of X is defined by the Lebesgue integral*

$$\mathbb{E}[X] := \int_\Omega \mathbf{X}(\omega)d\mathbb{P}(\omega).$$

In general

$$\mathbb{E}[g(x)] := \int_\Omega g(x)d\mathbb{P}.$$

We remark that if X *is a r.v whose CDF admits a pdf* $f(x)$ *, then* $\mathbb{E}[X]$ *is defined by the Lebesgue integral*

$$\mathbb{E}[X] = \int_\mathbb{R} xf(x)dx.$$

In general

$$\mathbb{E}[g(x)] = \int_\mathbb{R} g(x)f(x)dx.$$

Some desired properties of expectation are stated below.

1. **Constant preserved**
   If $X = c$, then $\mathbb{E}[X] = c$.

2. **Linearity**
   For $a,\ b \in \mathbb{R},\ \mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y]$.

3. **Monotonicity**
   If $X \leq Y,\ then\ \mathbb{E}[X] \leq \mathbb{E}[Y]$.

4. **Relation to the probability**
   For each event $A \subseteq \Omega,\ \mathbb{E}[\mathbf{1}] = \mathbb{P}(A)$.

Definition 17. *[3][**Conditional Density**]*
For any two events A and B

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}, \quad \mathbb{P}(B) \neq 0.$$

Definition 18. *[3][**Independent random variables**]*
Two events A and B are called independent if

$$\mathbb{P}(A \bigcap B) = \mathbb{P}(A)\mathbb{P}(B).$$

In other words, $\mathbb{P}(A|B) = \mathbb{P}(A)$.

Definition 19. *[3] Let $A_1, ..., A_n, ...$ be events. These events are independent if for all choices $1 \leq k_1 < k_2 < ... < k_m$, we have*

$$\mathbb{P}(A_{k_1} \bigcap A_{k_2} \bigcap ... \bigcap A_{k_m}) = \mathbb{P}(A_{k_1})\mathbb{P}(A_{k_2})....\mathbb{P}(A_{k_m}).$$

Theorem 1.2.1. *[3] If $X_1, ..., X_m$ are independent, real- valued random variables, with*
$\mathbb{E}[|X_i|] < \infty \qquad (i = 1, ..., m)$,
then

$$\mathbb{E}[(X_1...X_m)] = \mathbb{E}[X_1]...\mathbb{E}[X_m].$$

and $\mathbb{E}[|X_1...X_m|] < \infty$.

Definition 20. *[3][**Filtration**]*
A $filtration$ on a measurable space $(\Omega, \mathcal{F})$ is a nondecreasing family $\{\mathcal{F}_t, t \in T\}$ of sub σ -algebra s.t $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

So the filtration $\mathcal{F}_t$ can be interpreted as representing all historical information available to the economic agent who is observing the financial market up to time t. but not future information.

Definition 21. *[1][**Augmented Filtration**]*
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, let $\{\mathcal{F}_t\}_{t \in T}$ be an arbitrary filtration and let $\mathcal{N}$ be the collection of all negligible events in $\mathcal{F}$ that is $\mathcal{N} = \{A \in \mathcal{F}, \mathbb{P}(A) = 0\}$, then the filtration generated by $\mathcal{F} \bigcup \mathcal{N}$ given by:

$$\mathbb{F}_t = \sigma(\mathcal{F}_s \bigcup \mathcal{N}, s < t)$$

is called the augmented filtration or the augmentation of filtration $\mathcal{F}_t$.

Now we can introduce the definition of the adapted stochastic process. To proceed let T be subset of $\mathbb{R}$.

Definition 22. *[3][**Adapted Process**]*
Let $X = \{X_t, t \in T\}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then X is said to be adapted to the filtration $\{\mathcal{F}_t\}_{t \in T}$ if for every $t \in T, X_t$ is $\mathcal{F}_t$ -measurable.

For continuous-time processes, where the time t ranges over an arbitrary set T subset of $\mathbb{R}$, the property of being adapted is not enough. So, we consider the following process.

Definition 23. *[28][**Progressively Measurable Process**]*
The process $\{X_t, t \in T\}$ is said to be progressively measurable with respect to the filtration $\{\mathcal{F}_t, t \in T\}$, if for all $t \in T$, the mapping $(s, \omega) \in [0,t] \times \Omega \to X(s, \omega)$ is $(\mathcal{B}[0,t] \otimes \mathcal{F}_t) - measurable$.

Below we are going to talk about an important example of stochastic processes which is called *Brownian motion*.

Definition 24. *[18][**Brownian Motion**]*
The standard-one-dimensional Brownian motion BM or which is called sometimes the Weiner process is a real-valued stochastic process $W = \{W_t, t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ adapted to filtration $\mathcal{F}_t$ which satisfies the following properties:

1. *$W(0) = 0$ a.s,*
2. *$W(t) - W(s)$ is $N(0, t-s)$ for all $t \geq s \geq 0$,*
3. *for all $0 < t_1 < t_2 < ... < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), ..., W(t_n) - W(t_{n-1})$ are independent ("independent increments").*

We now introduce some remarks:

1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. An event $E \in \mathcal{F}$ happens almost surely (a.s), if $\mathbb{P}(E) = 1$.
2. A property which is true except for an event of probability zero is said to hold almost everywhere (abbreviated "a.e").
3. If W is a Brownian motion, then $\mathbb{E}[W(t)] = 0$ and $\mathbb{E}[W^2(t)] = t$ for each time $t \geq 0$. In other words

$$P(a \leq W(t) \leq b) = \frac{1}{\sqrt{2\pi t}} \int_a^b e^{\frac{-x^2}{2t}} dx.$$

4. The continuous sample path $t \to W(t,\omega),\ t \geq 0$ is nowhere differentiable a.s.

Below is a graph of the stochastic process BM which will illustrate some of its properties.

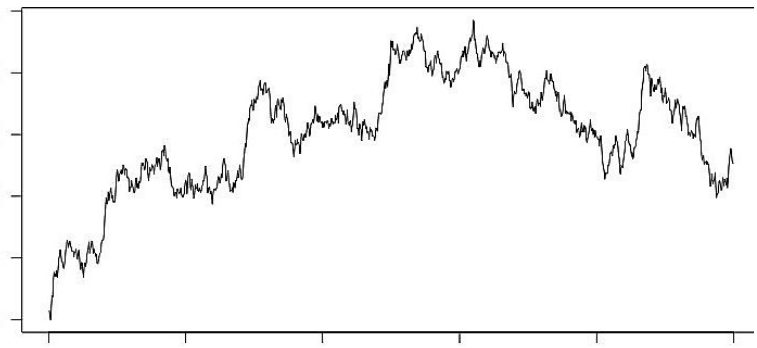

Figure 1.2: A Sample Path of Brownian Motion

1.3 Financial and life insurance notations

In this section, we introduce some important concepts concerning our work:

1. The discounted rate $\rho(t)$ is the interest charged by the central bank on a certain bank when taking a loan for example.
2. The appreciation rate $\mu(t)$ can be interpreted as the increase in the value of an asset over time.
3. The interest rate $r(t)$ is the amount that the lender charged to a borrower as a percentage for the use of an assets.
4. The volatility $\sigma(t)$ is a statistical measure that indicates the dispersion of returns for a given security. It can be measured either by using the standard deviation or variance between returns. So higher the volatility, higher the risk of the security.
5. The hazard function $\lambda(t)$ is a function that indicates the measure of risk, greater $\lambda(t)$, higher the risk.
6. The risk premium $\alpha(t)$ is a kind of compensation for investors who tolerate the extra risk compared to that of risk-free asset.

7. Life insurance is a contract or a deal that provides as a compensation for the family or some other named beneficiaries after insurer's death.
8. The insurance company premium-payout ratio $\eta(t)$ is the proportion of earnings paid out as dividends to shareholders, typically , it is expressed as a percentage.
9. Premium insurance rate $p(t)$ is the amount of money that an individual or business had to pay for an insurance policy.
10. The ansatz function is a starting function to solve a second order non-linear partial differential equation.
11. The utility function is an indicator of preferences or happiness of an economic-agent over some goods or services.
12. Constant Relative Risk Aversion CRRA of the utility function U means that $U$is an increasing and strictly concave function with the property that $-c\frac{U''(c)}{U'(c)}$ is constant.
13. The white noise ξ is the derivative of the Wiener process $W(t)$; i.e,

$$\xi(t) = \dot{W}(t) = \frac{dW(t)}{dt},$$

which does not really exists.

1.4 Examples of Stochastic Differential Equations

In this section, we are going to discuss the meaning of stochastic differential equations and the difference with the ordinary differential equations.

Let us first consider the ordinary differential equation (ODE) in the following example.

Example 4. *Solve the following ODE*

$$\frac{dX}{dt} = 2(t^2+1)X, \qquad X(0) = 1, \qquad X > 0. \tag{1.1}$$

Solution. The differential equation (1.1) can be written as

$$\frac{1}{2X}dX = (t^2+1)dt.$$

Integrating the left side with respect to X and the right side with respect to t gives

$$\ln X = \frac{2}{3}t^3 + 2t + c.$$

Thus,

$$X(t) = Ce^{\frac{2}{3}t^3+2t},$$

where C is an arbitrary constant. Applying the initial condition $X(0) = 1$ gives $C = 1$. Thus, the solution of (1.1) is given by

$$X(t) = e^{\frac{2}{3}t^3+2t}$$

■

However the experimentally measured trajectories of systems in many applications modeled by ODE do not behave as expected. Therefore, it is reasonable to modify the ODE into another model that includes the randomness as below.

$$\dot{\mathbf{X}}(t) = b(\mathbf{X}(t)) + B(\mathbf{X}(t))\xi, \qquad X(0) = x_0, \tag{1.2}$$

where $B : \mathbb{R}^n \to \mathbb{M}^{n\times m}(space\ of\ n \times m\ matrices)$ and

$$\xi(\cdot) := m - dimensional\text{“}white\ noise\text{”}.$$

And the trajectory $X : [0, \infty) \to \mathbb{R}^n$ is called the state of the system. Now write equation (1.2) as $\frac{d}{dt}$ instead of the dot as below:

$$\frac{d\mathbf{X}(t)}{dt} = b(\mathbf{X}(t)) + B(\mathbf{X}(t))\frac{d\mathbf{W}(t)}{dt}.$$

Multiplying by “dt”:

$$d\mathbf{X}(t) = b(\mathbf{X}(t))dt + B(\mathbf{X}(t))d\mathbf{W}(t), \qquad \mathbf{X}(0) = x_0.$$

The above equation is interpreted as a stochastic differential equation(SDE). We say that $\mathbf{X}(\cdot)$ solves the (SDE) provided that

$$\mathbf{X}(t) = x_0 + \int_0^t b(\mathbf{X}(s))ds + \int_0^t B(\mathbf{X}(s))d\mathbf{W}, \quad \forall\ t > 0.$$

We say that a stochastic differential equation abbreviated SDE is a differential equation in which one or more of the terms is a stochastic process. Moreover, the solution of this SDE is also a stochastic process. Typically, SDE contains a variable which represents a random variable. Now we give some examples of SDEs:

Example 5. *[3][**Optimal Portfolio Problem**]*
Suppose that a person has two investment possibilities:

- *A safe investment (e.g, a bond), where the price $X_0(t)$ per unit at time t grows exponentially:*

$$\frac{dX_0}{dt} = \rho X_0, \qquad \textit{where } \rho > 0 \textit{ is a constant.}$$

- *A risky investment (e.g, a stock), where the price $X_1(t)$ per unit at time satisfies a stochastic differential equation*

$$\frac{dX_1}{dt} = (\mu + \sigma.\text{“}noise\text{”})X_1,$$

where $\mu > \rho$ and $\sigma \in \mathbb{R} \setminus \{0\}$ are constants.

At each instant t, the person can choose how large portion(fraction) θ_t of his fortune Z_t he wants to place in the risky investment, thereby placing $(1-\theta_t)Z_t$ in the safe investment. Given a utility function U and a terminal time T the problem is to find the optimal portfolio $\theta_t \in [0,1]$, which maximize his expected utility at the corresponding terminal fortune $Z_T^{(\theta)}$; i.e,

$$\max\{E[U(Z_T^{(\theta)})]\}.$$

We now introduce an auxiliary result that play an important role in the proof of the main results.

Theorem 1.4.1 (Ito's Formula). *[3]. Suppose that $X(\cdot)$ solves the stochastic differential*

$$dX = Fdt + GdW.$$

Assume $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is continuous and that $\frac{\partial u}{\partial t}$, $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$ exists and are continuous. Let

$$Y(t) = u(X(t), t).$$

Then, Y solves the stochastic differential equation

$$\begin{aligned} dY &= \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dX + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2 dt, \\ &= \left(\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}G^2\right)dt + \frac{\partial u}{\partial x}GdW. \qquad (1.3) \end{aligned}$$

We call the stochastic differential equation (1.3) by the Ito's formula or Ito's chain rule. That is, for all $0 \leq s \leq r \leq T$, we have

$$\begin{aligned} Y(r) - Y(s) &= u(X(r), r) - u(X(s), s), \\ &= \int_s^r \frac{\partial u}{\partial t}(X,t) + \frac{\partial u}{\partial x}(X,t)F + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(X,t)G^2 dt \\ &+ \int_s^r \frac{\partial u}{\partial x}(X,t)GdW. \end{aligned}$$

Proof. See [18] for the proof. □

Example 6. *[3] A solution to stochastic differential equation of the form*

$$dX = D(t)Xdt + Q(t)XdW, \qquad X(0) = 1, \tag{1.4}$$

is given by

$$X(t) = e^{\int_0^t Q(s)dW(s) + \int_0^t (D(s) - \frac{1}{2}Q^2 s)ds}$$

To show that $X(t)$ is a solution for the SDE (1.4), we use Theorem 1.4.1. Note that

$$Y(t) = \int_0^t Q(s)dW(s) + \int_0^t (D(s) - \frac{1}{2}Q^2(s))ds$$

satisfies

$$dY = Q(t)dW(t) + (D(t) - \frac{1}{2}Q^2(t))dt.$$

Thus, using Itô's formula with $u(y) = e^y$ we get

$$\begin{aligned} dX &= \frac{\partial u}{\partial y}dY + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}Q^2 dt \\ &= e^Y\left(D(t)dt + \frac{-1}{2}Q^2 dt + Q(t)dW + \frac{1}{2}Q^2 dt\right) \\ &= D(t)Xdt + Q(t)XdW. \end{aligned}$$

Note that the initial condition is satisfied.

Example 7. *[3] Consider the following stochastic differential equation:*

$$dY = YdW, \qquad Y(0) = 1,$$

whose solution is given by

$$Y(t) = e^{W(t) - \frac{t}{2}}.$$

The proof of this special case follows similarly to Example 6.

Chapter 2

Optimal life-insurance selection and purchase within a market of several life-insurance providers

In this chapter, we review the work of Mousa et al paper [13]. In their paper, the authors considered the case of an economic- agent whose lifetime is uncertain and facing the problem of optimizing his decisions regarding consumption, investment and life- insurance selection and purchase during a random interval of time $[0, \min\{\tau, T\}]$, where T is a fixed instant of time in the future that can be seen as the retirement time of the economic-agent, and τ is a continuous and non-negative random variable representing the economic-agent's eventual time of death. The authors assume that the economic-agent observes the financial market and invests the full amount of his savings in that market which consists of one risk-free asset and $N \geq 1$ risky assets whose prices are $S_n(t),\ n = 0, 1, ..., N$. They also assume that the economic-agent is interested in buying life insurance in order to protect his family against the premature of his death from a market that has K life-insurance companies competing in this market. And when the economic-agent buys life-insurance from the k^{th} company, he has to pay a premium $p_k(t)$ to that insurance company.

2.1 Problem formulation

In this section, we introduce the financial market model available to the economic-agent, the life insurance market model, the wealth process, the optimal control problem and finally the stochastic optimal control problem.

2.1.1 Financial market model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ given by the $\mathbb{P}$ -augmentation of the filtration generated by M-dimensional Brownian motion $W(\cdot)$, $\sigma(W(s) : s \leq t)$ for $t \geq 0$, which means the history of the BM up to (including) time t, and $\mathcal{F}_t$ represents the information given to any agent observing the financial market during the time interval $[0, t]$.
Now let us consider a financial market consisting of one risk-free asset and $N \geq 1$ risky-assets. Their respective prices $(S_0(t))_{0\leq t\leq T}$ and $(S_n(t))_{0\leq t\leq T}$, for $n = 1, ..., N$ are given by

$$
\begin{aligned}
dS_0(t) &= r(t)S_0(t)dt, \quad S_0(0) = s_0,\\
dS_n(t) &= \mu_n(t)S_n(t)dt + S_n(t)\sum_{n=1}^{N}\sum_{m=1}^{M}\sigma_{nm}(t)(t)dW_n(t), \quad S_n(0) = s_n,
\end{aligned}
$$

where $W(t) = (W_1(t), ..., W_M(t))^T \in \mathbb{R}^M$ is the M-dimensional Brownian motion, $r(t)$ is the riskless interest rate, $\mu(t) = (\mu_1(t), ..., \mu_N(t))^T \in \mathbb{R}^N$ is the risky-assets appreciation rates vector, $\sigma(t) = (\sigma_{nm}(t)), 1 \leq n \leq N,\ 1 \leq m \leq M$ is the $N \times M$ matrix of risky-assets volatilities.

Assumption 1. *[13] The coefficients $r(t)$, $\mu(t)$ and $\sigma(t)$ are deterministic continuous functions on the interval $[0, T]$. In addition to the following conditions:*

1. *the interest rate $r(t) > 0, \quad \forall\, t \in [0, T]$;*
2. *the matrix $\sigma(t)$ is such that $\sigma(t)(\sigma(t))^T$ is non singular for Lebesgue almost all $t \in [0, T]$ and satisfies the following integrability condition*

$$\sum_{n=1}^{N}\sum_{m=1}^{M}\int_0^T \sigma^2_{nm}(t) < \infty;$$

3. *there exists an $(\mathcal{F}_t)_{0\leq t\leq T}$ -progressively measurable process $\pi(t) \in \mathbb{R}^M$, called the market price of risk, such that for Lebesgue-almost-every $t \in [0, T]$, the risk premium*

$$\alpha(t) = (\mu_1(t) - r(t), ..., \mu_N(t) - r(t))^T \in \mathbb{R}^N, \qquad (2.1)$$

is related to π *by the equation*

$$\alpha(t) = \sigma(t)\pi(t) \ a.s.$$

Also

$$\int_0^T \|\pi(t)\|^2 dt < \infty \quad a.s,$$

$$\mathbb{E}[e^{-\int_0^T \pi dW - \frac{1}{2}\int_0^T \|\pi(t)\|^2 dt}] = 1,$$

where $\|.\|$ *denotes the Euclidean norm in* $\mathbb{R}^n$*.*

2.1.2 Life-insurance market model

Assumption 2. *Assume that the economic-agent is alive at time* $t = 0$ *with uncertain lifetime given by a continuous non-negative random variable* τ *defined on the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*.*

Assumption 3. *[13] The random variable* τ *is independent of the filtration* $\mathbb{F}$ *with distribution function* $G^- : [0, \infty) \to [0, 1]$ *and density* $g^- : [0, \infty) \to \mathbb{R}^+$ *such that*

$$G^-(t) = p(\tau \le t) = \int_0^t g^-(s) ds.$$

Note that the survival function $G^+ : [0, \infty) \to [0, 1]$ *is defined as the probability for the economic-agent to survive past time* t*; i.e.*

$$G^+(t) = p(\tau > t) = 1 - G^-(t).$$

The hazard rate function, the conditional, instantaneous death rate for the economic-agent surviving past time t*, given by*

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t < \tau \le t + \Delta t | \tau > t)}{\Delta t}. \tag{2.2}$$

Note that from identity (2.2) *and the definition of the density function we have*

$$\begin{aligned} \lambda(t) &= \lim_{\Delta t \to 0} \frac{P(\tau \le t + \Delta t) - P(\tau \le t)}{\Delta t \ P(\tau > t)}, \\ &= \lim_{\Delta t \to 0} \frac{G^-(t + \Delta t) - G^-(t)}{\Delta t P(\tau > t)}, \\ \textit{Thus,} \quad \lambda(t) &= \frac{g^-(t)}{G^+(t)}. \end{aligned}$$

It then follows that

$$\lambda(t) \quad = \quad -\frac{d}{dt}(\ln G^{+}(t)).$$

Hence the survivor function can be written as

$$G^{+}(t) \quad = \quad e^{-\int_0^t \lambda(u)du},$$

and the probability density function is related to the hazard rate by

$$g^{-}(t) \quad = \quad \lambda(t)e^{-\int_0^t \lambda(u)du}.$$

Suppose that the hazard rate function $\lambda : [0,\infty) \to \mathbb{R}^{+}$ *is a deterministic continuous function such that*

$$\int_0^{\infty} \lambda(t)dt = \infty.$$

The concepts introduced above are standard in the context of Reliability Theory and Actuarial Science see [15], [25].

If the economic-agent dies at time $\tau \leq T$ *while having a contract with the* k^{th} *insurance company by buying insurance at rate* $p_k(t)$*, then that insurance company pays an amount*

$$Z_k(\tau) = \frac{p_k(\tau)}{\eta_k(\tau)},$$

to his estate, where $\eta_k : [0,T] \to \mathbb{R}^{+}$ is the k^{th} insurance company premium-payout ratio. Such ratio determines the life-insurance payout in the event of death and it is fixed by the insurance company.

The contract ends when the economic-agent dies or achieves retirement age, whichever happens first. So, the economic-agent's total legacy to his estate in the event of a premature death at time $\tau \leq T$ is given by

$$Z(\tau) = X(\tau) + \sum_{k=1}^{k} \frac{p_k(\tau)}{\eta_k(\tau)}, \tag{2.3}$$

where $X(t)$ denotes the economic-agent's wealth at time $t \in [0,T]$.

Assumption 4. *[13] For every $k \in \{1, ..., K\}$, the k^{th} insurance company premium-payout ratio $\eta_k(t)$ is deterministic and contiuous function. Additionally, we also assume that the K insurance companies in our model offer pairwise distinct contracts such that $\eta_{k_1}(t) \neq \eta_{k_2}(t)$, $\forall\ k_1 \neq k_2$ and Lebesgue almost every $t \in [0, T]$.*

The economic-agent life- insurance purchase rate as a vector is given by

$$p(t) = (p_1(t), p_2(t), ..., p_k(t))^T \in (\mathbb{R}_0^+)^k,$$

where for each $k \in \{1, 2, ..., K\}$, the quantity $p_k(t)$ denotes the life-insurance rate paid to the k^{th} insurance company at time $t \in [0, \min\{\tau, T\}]$. Note that the zero component in $p(t)$ represents that there is no any life-insurance contract between the economic-agent and that insurance company.

2.1.3 Wealth process

The economic-agent gets income $i(t)$ at a continuous rate during the period $[0, \min\{\tau, T\}]$. That is, the income will be terminated by his death or his retirement, whichever happens first.

Assumption 5. *[13] The income function $i : [0, T] \to \mathbb{R}_0^+$ is a deterministic Borel-measurable satisfying the integrability condition:*

$$\int_0^T i(t)dt < \infty.$$

The consumption process $(c(t))_{0 \leq t \leq T}$ is a $(\mathcal{F}_t)_{0 \leq t \leq T}$ -progressively measurable non- negative process satisfying the following integrability condition for the investment horizon $T > 0$:

$$\int_0^T c(t)dt < \infty \ a.s.$$

We assume also that for all k=1,2,...,K, the kth company premium insurance rate$(p_k(t))_{0 \leq t \leq T}$ is a non-negative $(\mathcal{F}_t)_{0 \leq t \leq T}$ - measurable process with respect to the smallest $\sigma-$ algebra on $\mathbb{R}_0^+ \times \Omega$.
For each $n = 0, 1, ..., N$ and $t \in [0, T]$, let $\theta_n(t)$ denote the fraction of the economic-agent's wealth allocated to the asset S_n at time t . The economic-agent portfolio process is then given by $\Theta(t) = (\theta_0(t), \theta_1(t), ..., \theta_N(t))^T \in \mathbb{R}^{N+1}$, where

$$\sum_{n=0}^{N} \theta_n(t) = 1, \qquad 0 \leq t \leq T.$$

The portfolio process is assumed to be $(\mathcal{F}_t)_{0\leq t\leq T}$ -progressively measurable and that for the fixed investment horizon $T>0$ we have that

$$\int_0^T \|\Theta\|^2 dt < \infty \qquad a.s.,$$

where $\|.\|$ denotes the Euclidean norm in $\mathbb{R}^{N+1}$.

The wealth process $X(t),\ t\in[0,\min\{\tau,T\}],$ is then defined by

$$X(t) = x_0 + \int_0^t \Big(i(s) - c(s) - \sum_{k=1}^{K} p_k(s)\Big)ds + \sum_{n=0}^{N}\int_0^t \frac{\theta_n(s)X(s)}{S_n(s)}dS_n(s), \quad (2.4)$$

where x_0 is the economic-agent's initial wealth.

Differentiate equation (2.4) with respect to t then it can be rewritten in the differential form as

$$dX(t) = \Big(i(t) - c(t) - \sum_{k=1}^{K} p_k(t) + \Big(\theta_0(t)r(t) + \sum_{n=1}^{N}\theta_n\mu_n(t)\Big)X(t)\Big)\Big)dt$$

$$+ \sum_{n=1}^{N}\theta_n(t)X(t)\sum_{m=1}^{M}\sigma_{nm}(t)dW_m(t), \quad (2.5)$$

where $0\leq t\leq \min\{\tau,T\}$. Since $\theta_0+\theta_1+...+\theta_N=1,$ the reduced portfolio process $\theta(t)\in\mathbb{R}^N$, is given by

$$\theta(t) = (\theta_1(t),\theta_2(t),...,\theta_N(t))^T \in \mathbb{R}^N.$$

2.2 Optimal control problem

Now the economic-agent wants to maximize the expected utility obtained from: his family consumption for all $t\leq\min\{\tau,T\}$; his wealth at retirement date T if he lives that long; and the value of his estate in the event of premature death.

Denote by $\mathcal{A}(0,x)$ the set of all admissible decision strategies which is the set of all 3-tuples $(c(\cdot),p(\cdot),\theta(\cdot))$ such that $X(t)+b(t)\geq 0$ and $Z(t)\geq 0$, $\forall\ t\ \in[0,T]$ with the boundary condition $X(0)=x_0$. Where $X(t)$, $Z(t)$ are given in (2.4), (2.3) respectively and $b(t)$ is given in (2.27).

The economic-agents's problem then is to find a strategy $v=(c(\cdot),\theta(\cdot),p(\cdot))\in\mathcal{A}(0,x)$ which maximize the expected utility

$$\mathbb{E}_{0,x}\Big[\int_0^{\tau\wedge T} U(s,c(s))ds + B(\tau,Z(\tau))\mathbf{1}_{[0,T]}(\tau) + W(X(T))\mathbf{1}_{(T,\infty)}(\tau)\Big],$$

where $\tau \wedge T = \min\{\tau, T\}$, $\mathbf{1}_A$ denotes the indicator function of the set A, U is the utility function for the economic-agent's family consumption at some instant of time $t \in [0, T]$, W is the utility function for the terminal wealth at retirement time T, *and* B is the utility function for the size of the economic-agent's legacy at time $t \in [0, T]$.

Assumption 6. *[13] The utility functions* $U : [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$, *and* $B : [0, T] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *are twice differentiable, strictly increasing and strictly concave functions on their second variable, and* $W : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is a twice differentiable, strictly increasing and strictly concave function.*

2.3 Stochastic optimal control problem

In this section we refer to the paper introduced by Ye [10] to formulate the stochastic optimal control problem to one with a fixed planning horizon. We then state a dynamic programming principle and its corresponding HJB equation.

Let $v = (c(\cdot), \theta(\cdot), p(\cdot))$ be the decision strategies for the dynamics of the wealth process with boundary condition $X(t) = x$. For any $v \in \mathcal{A}(t, x)$, we define

$$J(t, x; v) = \mathbb{E}_{t,x}\Bigg[\int_t^{\tau \wedge T} U(s, c(s))ds + B(\tau, Z(\tau))\mathbf{1}_{[0,T]}(\tau) \qquad (2.6)$$

$$+W(X_{t,x}^v(T))\mathbf{1}_{(T,+\infty)}(\tau)|\tau > t, \mathcal{F}_t\Bigg],$$

where $X_{t,x}^v(s) \geq 0$ denotes the wealth process starting from x at time $t \leq s$ under the selection of the control $v \in \mathcal{A}(t, x)$. That is, $X_{t,x}^v(s)$ is the solution of the stochastic differential equation (2.5) with initial condition $X(t) = x$. $\forall\ t \in [0, s]$, let $G^+(s, t)$ the conditional probability for the economic-agent to be alive at time s conditional upon being alive at $t \leq s$; i.e,

$$\begin{aligned} G^+(s, t) &= P\Big(\{\tau > s\}|\{\tau > t\}\Big) \\ &= \frac{\mathbb{P}(\tau > s)}{\mathbb{P}(\tau > t)} \\ &= \frac{G^+(s)}{G^+(t)}. \end{aligned}$$

Thus,

$$G^+(s, t) = e^{-\int_t^s \lambda(u)du}. \qquad (2.7)$$

Also $G^-(s,t)$ denote the conditional probability for the economic- agent time of death to occur at time s conditional upon being alive at time $t \leq s$, given by

$$G^-(s,t) = P\Big(\{\tau \leq s\}|\{\tau > t\}\Big).$$

Let $g^-(s,t)$ be the density function corresponds to $G^-(s,t)$ that is,

$$g^-(s,t) = \frac{d}{ds}G^-(s,t).$$

In other words

$$\begin{aligned} g^-(s,t) &= \frac{d}{ds}(1 - G^+(s,t)) \\ &= -\frac{d}{ds}G^+(s,t) \\ &= -\frac{d}{ds}\frac{G^+(s)}{G^+(t)} \\ &= \frac{g^-(s)}{G^+(t)}. \end{aligned}$$

Thus,

$$g^-(s,t) = \lambda(s)e^{-\int_t^s \lambda(u)du}. \tag{2.8}$$

In the next result, we rewrite the optimal control problem with a fixed horizon T.

LEMMA 2.3.1. *[13] Assume that the Assumptions above are satisfied. If the random variable τ is independent of the filtration $\mathbb{F}$, then*

$$J(t,x;v) = \mathbb{E}_{t,x}\Bigg[\int_t^T \Big(G^+(s,t)U(s,c(s)) + g^-(s,t)B(s,Z(s))\Big)ds$$
$$+G^+(T,t)W(X(T))|\mathcal{F}_t\Bigg].$$

Proof. From (2.6), we can rewrite J as

$$J(t,x;c,p,\theta) = \mathbb{E}\Bigg[\mathbf{1}_{\{\tau \geq T\}}\int_t^T U(c(s),s)ds + \mathbf{1}_{\{\tau < T\}}\int_t^\tau U(c(s),s)ds$$
$$+B(Z(\tau),\tau)\mathbf{1}_{\{\tau \leq T\}} + W(X(T))\mathbf{1}_{\{\tau > T\}} \mid \tau > t, \mathcal{F}_t\Bigg].$$

Note that $g^{-}(u,t),\ u \geq t$ is the conditional probability density of τ and τ is independent of the filtration $\mathbb{F}$. Hence we have that

$$J(t,x;c,p,\theta) = \mathbb{E}\Big[G^{+}(T,t)\int_t^T U(c(s),s)ds + \int_t^T g^{-}(u,t)\int_t^u U(c(s),s)dsdu$$

$$+ \int_t^T g^{-}(u,t)B(Z(u),u)du + W(X(T))G^{+}(T,t) \mid \mathcal{F}_t\Big]. \qquad (2.9)$$

Because $g^{-}(u,t)U(s,c(s))$ is nonnegative, we can use the Fubini-Tonelli theorem, so the order of the integration can be interchanged, i.e

$$\int_t^T \int_t^u g^{-}(u,t)U(s,c(s))dsdu$$

$$\begin{aligned} &= \int_t^T \int_s^T g^{-}(u,t)U(s.c(s))duds \\ &= \int_t^T \Big(\int_s^T g^{-}(u,t)du\Big)U(s,c(s))ds \\ &= \int_t^T (G^{+}(s,t) - G^{+}(T,t))U(u,c(u))du. \end{aligned} \qquad (2.10)$$

Hence by (2.9) and (2.10) we get our result. □

After we transformed the optimal control problem using fixed T, we can now use the idea of dynamic programming. To proceed, let

$$V(t,x) = \sup_{v\in\mathcal{A}(t,x)} J(t,x;v).$$

From Lemma 2.3.1, we now state a dynamic programming principle(DPP) to get a recursive definition of the value function $V(t,x)$. But before let us introduce the following definition that will help us stating the next Lemma.

Definition 25. *[12] Let $X(\cdot)$ is a continuous stochastic process defined on the filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\})$, then $X(\cdot)$ has a Markov property if for $0 < t < s < T$, we have*

$$\mathbb{E}[X_s \mid \mathcal{F}_t] = \mathbb{E}[X_s \mid X_t].$$

Markov property or sometimes called memoryless property can be explained as a property in which its behaviour is not affected by the past but depends only on the present state, for example, the stochastic process of the observed color while drawing a ball from a box is considered to be Markovian if it was with replacement and non-markovian if without replacement (the present value is affected by the information about the past).

LEMMA 2.3.2. *[13] (DPP). Suppose that all Assumptions above are hold. For $0 \leq t < s < T$, then the maximum expected utility $V(t,x)$ satisfies the recursive relation*

$$V(t,x) = \sup_{v \in \mathcal{A}(t,x)} \mathbb{E}\Big[exp\Big(- \int_t^s \lambda(u)du\Big) V(s, X_{t,x}^v(s))$$

$$+ \int_t^s \Big(G^+(u,t)U(u,c(u)) + g^-(u,t)B(u, Z_{t,x}^v(u))\Big) du | \mathcal{F}_t\Big].$$

Proof. For any $(c(\cdot), p(\cdot), \theta(\cdot)) \in \mathcal{A}(t,x)$ with the corresponding wealth $X_{t,x}^{c,p,\theta}(\cdot)$ and the corresponding legacy $Z_{t,x}^{c,p,\theta}(\cdot)$, according to Lemma 2.3.1,

$$\begin{aligned}
J(t,x,c(\cdot),p(\cdot),\theta(\cdot)) &= \mathbb{E}\Big[\int_t^T \Big(g^-(u,t)B(u, Z_{t,x}^{c,p,\theta}(u)) + G^+(u,t)U(u,c(u))\Big)du \\
&\qquad + G^+(T,t)W(X(T)) \mid \mathcal{F}_t\Big] \\
&= \mathbb{E}\Big[\int_s^T \Big(g^-(u,t)B(u, Z_{t,x}^{c,p,\theta}(u)) + G^+(u,t)U(u,c(u))\Big)du \\
&\quad + \Big(\int_t^s \Big(g^-(u,t)B(u, Z_{t,x}^{c,p,\theta}(u)) + G^+(u,t)U(u,c(u))\Big)du \\
&\qquad + G^+(T,t)W(X(T)) \mid \mathcal{F}_t\Big].
\end{aligned} \tag{2.11}$$

Note that

$$\begin{aligned}
g^-(u,t) &= \lambda(u)e^{-\int_t^u \lambda(v)dv} \\
&= e^{-\int_t^s \lambda(v)dv}\lambda(u)e^{-\int_s^u \lambda(v)dv} \\
&= e^{-\int_t^s \lambda(v)dv} g^-(u,s).
\end{aligned}$$

Similarly,

$$G^+(u,t) = e^{-\int_t^s \lambda(v)dv} G^+(u,s).$$

Then from (2.11), we have

$$J(t,x;c(\cdot),p(\cdot),\theta(\cdot)) = \mathbb{E}\Big[e^{-\int_t^s \lambda(v)dv}\Big\{\int_s^T g^-(u,s)B\Big(Z_{t,x}^{c,p,\theta}(u), u\Big)$$

$$+G^{+}(u,s)U(c(u),u)du\Big\} + G^{+}(T,s)W\Big(X_{t,x}^{c,p,\theta}(T)\Big)$$

$$+\int_t^s g^{-}(u,t)B\Big(u, Z_{t,x}^{c,p,\theta}(u)\Big) + G^{+}(u,t)U(u,c(u))du \mid \mathcal{F}_t\Big]. \tag{2.12}$$

Note that $X(\cdot)$ has the markov property and its proof is in [18], in which case

$$\begin{aligned}\mathbb{E}\Big[W\Big(X_{t,x}^{c,p,\theta}(T)\Big) \mid \mathcal{F}_t\Big] &= \mathbb{E}\Big[W\Big(X_{t,x}^{c,p,\theta}(T)\Big) \mid \mathcal{F}_s \mid \mathcal{F}_t\Big] \\ &= \mathbb{E}\Big[W\Big(X_{s,X_{t,x}(s)}^{c,p,\theta}(T)\Big) \mid \mathcal{F}_t\Big],\end{aligned}$$

where $(c(\cdot),p(\cdot),\theta(\cdot))$ restricted to the interval $[s,T]$ is in $\mathcal{A}(s, X_{t,x}^{c,p,\theta}(s))$. Hence (2.12) becomes

$$J(t,x;c(\cdot),p(\cdot),\theta(\cdot)) = \mathbb{E}\Big[e^{-\int_t^s \lambda(v)dv} J\Big(s, X_{t,x}^{c,p,\theta}(s); c(\cdot),p(\cdot),\theta(\cdot)\Big)$$

$$+\int_t^s g^{-}(u,t)B\Big(u, Z_{t,x}^{c,p,\theta}(u)\Big) + G^{+}(u,t)U(u,c(u))du \mid \mathcal{F}_t\Big]$$

$$\leq \mathbb{E}\Big[e^{-\int_t^s \lambda(v)dv} V\Big(s, X_{t,x}^{c,p,\theta}(s)\Big) + \int_t^s g^{-}(u,t)B\Big(u, Z_{t,x}^{c,p,\theta}(u)\Big)$$

$$+G^{+}(u,t)U(u,c(u))du \mid \mathcal{F}_t\Big].$$

Note that $(c(\cdot),p(\cdot),\theta(\cdot))$ is arbitrary, it follows that

$$V(t,x) \leq \sup_{(c,p,\theta)\in\mathcal{A}(t,x)} \mathbb{E}\Big[e^{-\int_t^s \lambda(v)dv} V\Big(s, X_{t,x}^{c,p,\theta}(s)\Big)$$

$$+\int_t^s g^{-}(u,t)B\Big(u, Z_{t,x}^{c,p,\theta}(u)\Big) + G^{+}(u,t)U(u,c(u))du \mid \mathcal{F}_t\Big]. \tag{2.13}$$

Conversely given that $(c(\cdot),p(\cdot),\theta(\cdot)) \in \mathcal{A}(t,x)$, for $\epsilon > 0$ *and* $\omega \in \Omega$, using property of supremum there exists

$$\mathcal{L} \equiv (c_{\omega,\epsilon}(\cdot), p_{\omega,\epsilon}(\cdot), \theta_{\omega,\epsilon}(\cdot)) \in \mathcal{A}\Big(s, X_{t,x}^{c,p,\theta}(s,\omega)\Big),$$

where

$$J\Big(s, X_{t,x}^{c,p,\theta}(s); \mathcal{L}_{\omega,\epsilon}(\cdot)\Big) \geq V\Big(s, X_{t,x}^{c,p,\theta}(s)\Big) - \epsilon.$$

Let

$$\mathcal{L}^*(u) := \begin{cases} (c(u), p(u), \theta(u)) & \text{if } u \in [t, s] \\ \mathcal{L}_{\omega,\epsilon}(u) & \text{if } u \in [s, T]. \end{cases}$$

Notice that $X^{\mathcal{L}^*}_{t,x}(T) = X^{\mathcal{L}_{\omega,\epsilon}}_{s, X^{c,p,\theta}_{t,x}(s)}(T)$ almost surely. Thus from (2.12), we get

$$\begin{aligned} V(t,x) &\geq J(t,x;\mathcal{L}^*(\cdot)) \\ &= \mathbb{E}\Bigg[e^{-\int_t^s \lambda(v)dv}\Bigg\{\int_s^T g^-(u,s)B\Big(Z^{\mathcal{L}_{\omega,\epsilon}}_{t,x}(u), u\Big) \\ &\quad + G^+(u,s)U(c_{\omega,\epsilon}(u), u)du\Bigg\} + G^+(T,s)W\Big(X^{\mathcal{L}_{\omega,\epsilon}}_{s,X^{c,p,\theta}_{t,x}(s)}(T)\Big) \\ &\quad + \int_t^s g^-(u,t)B\Big(u, Z^{c,p,\theta}_{t,x}(u)\Big) + G^+(u,t)U(u,c(u))du \mid \mathcal{F}_t\Bigg]. \\ &\geq \mathbb{E}\Bigg[e^{-\int_t^s \lambda(v)dv}\Big(V\Big(t, X^{c,p,\theta}_{s,x}(t)\Big) - \epsilon\Big) \\ &\quad + \int_t^s g^-(u,t)B\Big(u, Z^{c,p,\theta}_{t,x}(u)\Big) + G^+(u,t)U(u,c(u))du \mid \mathcal{F}_t\Bigg]. \end{aligned}$$

Since this inequality holds for any $(c,p,\theta) \in \mathcal{A}(t,x)$ and $\epsilon > 0$, then

$$\begin{aligned} V(t,x) \geq \sup_{(c,p,\theta)\in\mathcal{A}(t,x)} &\mathbb{E}\Bigg[e^{-\int_t^s \lambda(v)dv}V\Big(s, X^{c,p,\theta}_{t,x}(s)\Big) \\ &+ \int_t^s g^-(u,t)B\Big(u, Z^{c,p,\theta}_{t,x}(u)\Big) + G^+(u,t)U(u,c(u))du \mid \mathcal{F}_t\Bigg]. \end{aligned} \tag{2.14}$$

Then the DPP follows from (2.13) and (2.14). □

DPP helps to write second-order nonlinear PDE whose solution is the value function of the optimal control problem.

Theorem 2.3.3. *[13] (Hamilton-Jacobi-Bellman Equation). Suppose that all the above Assumptions are hold and that the value function V is of class $C^{1,2}([0,T] \times \mathbb{R}, \mathbb{R})$. Then V satisfies the Hamilton- Jacobi- Bellman equation*

$$V_t(t,x) - \lambda(t)V(t,x) + \sup_{(c,\theta,p)\in\mathbb{R}^{N+1}\times(\mathbb{R}_0^+)^k} \mathcal{H}(t,x;c,\theta,p) = 0, \tag{2.15}$$

$$V(T,x) = W(x),$$

where the Hamiltonian function $\mathcal{H}$ is given by

$$\mathcal{H}(t,x;v) = \left(i(t) - c(t) - \sum_{k=1}^{K} p_k + \left(r(t) + \sum_{n=1}^{N} \theta_n(\mu_n(t) - r(t))\right)x\right)V_x(t,x) + \frac{x^2}{2}\sum_{m=1}^{M}\left(\sum_{n=1}^{N}\theta_n\sigma_{nm}(t)\right)^2 V_{xx}(t,x) + U(t,c) + \lambda(t)B\left(t, x + \sum_{k=1}^{K}\frac{p_k(T)}{\eta_k(t)}\right).$$

In addition $v^* = (c^*(\cdot), \theta^*(\cdot), p^*(\cdot)) \in \mathcal{A}(t,x)$ with wealth X^* is optimal if and only if for $t \in [t,T]$ we have

$$V_t(s, X^*(s)) - \lambda(s)V(s, X^*(s)) + \mathcal{H}(s, X^*(s); v^*) = 0.$$

The proof of Theorem 2.3.3 is similar to Theorem 3.4.4, so we skip it.

2.4 Optimal strategies

Theorem 2.3.3 gives us a strategy to compute the optimal insurance selection and purchase, portfolio and consumption strategies for the economic- agent with uncertain lifetime τ.

Let $U_x(t,\cdot)$ and $B_x(t,\cdot)$ denote, respectively, the derivative of the utility functions $U(t,\cdot)$ and $B(t,\cdot)$ with respect to their second arguments for each $t \in [0,T]$, . Since both $U(t,\cdot)$ and $B(t,\cdot)$ are strictly concave with respect to their second arguments, the corresponding derivatives are invertible. Hence, we can define $I_1 : [0,T] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and $I_2 : [0,T] \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ to be the (inverse) functions such that

$$I_1(t, U_x(t,x)) = x \quad \textit{and} \quad U_x(t, I_1(t,x)) = x, \tag{2.16}$$

and

$$I_2(t, B_x(t,x)) = x \quad \textit{and} \quad B_x(t, I_2(t,x)) = x, \tag{2.17}$$

for every $t \in [0,T]$ and $x \in \mathbb{R}_0^+$. The next result gives us the optimal strategies in terms of V, V_x, V_{xx}.

Theorem 2.4.1. *[13] Suppose Assumptions above are satisfied and $V \in C^{1,2}([0,T]\times \mathbb{R}, \mathbb{R})$. Then the Hamiltonian function $\mathcal{H}$ has a unique maximum $v^* = (c^*(\cdot), \theta^*(\cdot), p^*(\cdot)) \in \mathcal{A}(t,x)$. Moreover, the optimal strategies are given by*

$$c^*(t,x) = I_1(t, V_x(t,x)),$$

$$\theta^*(t,x) = -\frac{V_x(t,x)}{xV_{xx}(t,x)}\xi\alpha(t),$$

and, for each $k \in \{1, 2, ..., K\}$, we have that

$$p_k^*(t,x) = \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_k(t)V_x(t,x)}{\lambda(t)}\right) - x\right]\eta_k(t)\right\}, & if\ k = k^*(t) \\ 0, & Otherwise \end{cases}$$

where

$$k^*(t) = \underset{k\in\{1,2,...,K\}}{\operatorname{argmin}} \{\eta_k(t)\}, \tag{2.18}$$

and ξ *is a non-singular square matrix given by* $(\sigma\sigma^T)^{-1}$ *and* $\alpha(t)$ *is as given in* (2.1).

Proof. We are looking for $v^* = (c^*, \theta^*, p^*) \in \mathcal{A}(t,x)$ such that $\mathcal{H}$ attains its maximum value, note that the condition determining the maximum for $\mathcal{H}$ can be divided as follows:

$$\sup_{(c,\theta,p)\in\mathbb{R}^{N+1}\times(\mathbb{R}_0^+)^k} \mathcal{H}(t,x;c,\theta,p) = \sup_{c\in\mathbb{R}}\left\{U(t,c) - cV_x(t,x)\right\} + r(t)xV_x(t,x) +$$

$$\sup_{p\in(\mathbb{R}_0^+)^k}\left\{\lambda(t)B\left(t, x + \sum_{k=1}^K \frac{p_k(t)}{\eta_k(t)}\right) - V_x(t,x)\sum_{k=1}^K p_k\right\} + i(t)V_x(t,x)$$

$$+ \sup_{\theta\in\mathbb{R}^N}\left\{\frac{x^2}{2}\sum_{m=1}^M\left(\sum_{n=1}^N \theta_n\sigma_{nm}(t)\right)^2 \times V_{xx}(t,x) + \sum_{n=1}^N \theta_n\left(\mu_n(t) - r(t)\right)xV_x(t,x)\right\}.$$

Let us first deal with the unconstrained optimization problems and differentiate $\mathcal{H}$ with respect to c, θ respectively
First we differentiate with respect to c to obtain the condition

$$U_x(t,c) - V_x(t,x).$$

Differentiation with respect to θ of

$$\sum_{n=1}^N \theta_n(\mu_n(t) - r(t))xV_x(t,x) = (\mu_1 - r(t), ..., \mu_N - r(t))xV_x(t,x) = \alpha xV_x(t,x), \tag{2.19}$$

and the differentiation with respect to θ of

$$\frac{x^2}{2}\sum_{m=1}^M\left(\sum_{n=1}^N \theta_n\sigma_{nm}(t)\right)^2 \times V_{xx}(t,x) = x^2V_{xx}\sigma\sigma^T\theta. \tag{2.20}$$

Now combine (2.19) and (2.20) so the differentiation with respect to θ of

$$\frac{x^2}{2}\sum_{m=1}^M(\sum_{n=1}^N \theta_n\sigma_{nm}(t))^2 \times V_{xx}(t,x) + \sum_{n=1}^N \theta_n(\mu_n(t) - r(t))xV_x(t,x)$$

$$= x^2 V_{xx}\sigma\sigma^T\theta + \alpha x V_x(t,x).$$

So, the unconstrained optimization problems for c and θ is given by the following $N+1$ first-order conditions:

$$\begin{aligned} -V_x(t,x) + U_x(t,c^*) &= 0, & (2.21)\\ xV_x(t,x)\alpha + x^2 V_{xx}(t,x)\sigma\sigma^T\theta^* &= 0_{\mathbb{R}^N}, & (2.22)\end{aligned}$$

where $0_{\mathbb{R}^N}$ denotes the origin of $\mathbb{R}^N$. By using the inverse function introduced in (2.16) and solve equation (2.21) for the control variable c, we get:

$$I_1(t, U_x(t,c^*)) = I_1(t, V_x(t,x)),$$

$$c^*(t,x) = I_1(t, V_x(t,x)). \tag{2.23}$$

Solving equation (2.22) for the variable θ to obtain

$$\theta^*(t,x) = -\frac{V_x(t,x)}{xV_x(t,x)}\xi\alpha(t). \tag{2.24}$$

Now according to the constrained optimization problem associated with the variable $p \in (\mathbb{R}_0^+)^K$, we make use of Kuhn-Tucker conditions. In particular, we search for a solution $(p_1(t,x), ..., p_k(t,x), \mu_1(t,x), ..., \mu_k(t,x))$, with Lagrangian L given by:

$$L = \lambda(t) B\left(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}\right) - V_x(t,x)\sum_{k=1}^{K} p_k + p_k\mu_k.$$

Now if we differentiate L with respect to p_k, then by Kuhn-Tucker conditions we get the following set of equalities and inequalities:

$$\frac{\lambda(t)}{\eta_k(t)} B_x\left(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k(t)}\right) - V_x(t,x) = -\mu_k,$$

$$\begin{aligned} p_k &\geq 0,\\ \mu_k &\geq 0,\\ p_k\mu_k &= 0 \end{aligned}$$

for $k = 1, 2, ..., K$.

We have two cases either $p_k(t,x) = 0, \ \forall k \in \{1,2,...,K\}$ or $\exists k \in \{1,2,...,K\}$ such that $p_k(t,x) \neq 0$. Now assume $k_1 \neq k_2$ if $\mu_{k_1}(t) = \mu_{k_2}(t)$ for some $(t,x) \in [0,t] \times \mathbb{R}$ then we must have that $\eta_{k_1}(t) = \eta_{k_2}(t)$ and that contradicts our assumption that all insurance companies are pairwise distinct so we obtain for any $k_1, \ k_2 \in \{1,2,...,K\}$ such that $k_1 \neq k_2$ and every $x \in \mathbb{R}$, $\mu_{k_1}(t,x) \neq \mu_{k_2}(t,x)$ for Lebesgue a.e. $t \in [0,T]$. But the condition $p_k\mu_k = 0$ for all $k \in \{1,2,...,K\}$ so we conclude that at most $\exists k \in \{1,2,...,K\}$ such that $\mu_k(t,x) = 0$. And there is at most one $k \in \{1,2,..K\}$ such that $p_k(t,x) \neq 0$.
We also note that

$$\begin{aligned} I_2(t, B_x(t, x + \sum_{k=1}^{K} \frac{p_k}{\eta_k})) &= I_2\Big(t, (V_x(t,x) - \mu_{k_1})\Big)\frac{\eta_{k_1}(t)}{\lambda(t)} \\ &= I_2\Big(t, (V_x(t,x) - \mu_{k_2})\Big)\frac{\eta_{k_2}(t)}{\lambda(t)}. \end{aligned}$$

So, we get the identity below

$$\eta_{k_1}(t)(V_x(t,x) - \mu_{k_1}) = \eta_{k_2}(t)(V_x(t,x) - \mu_{k_2}).$$

As a consequence of the identity above, we conclude that if $\mu_{k_1}(t,x) > \mu_{k_2}(t,x)$ for $(t,x) \in [0,T] \times \mathbb{R}$, then $\eta_{k_1} > \eta_{k_2}$. Furthermore, if for some $t \in [0,T]$ we have $\mu_{k_1}(t,x) = 0$, then $\eta_{k_1} < \eta_{k_2}$. for every $k_2 \in \{1,2,...,K\}$ such that $k_1 \neq k_2$.

Thus, let $k^*(t)$ be as given in (2.18) . Then, either $p_k(t,x) = 0$, for every $k \in \{1,2,...,K\}$ or else $p_{k^*(t)} > 0$ is a solution to

$$\frac{\lambda(t)}{\eta_{k^*}(t)} B_x\left(t, x + \frac{p_{k^*}(t)}{\eta_{k^*}(t)}\right) = V_x(t,x). \tag{2.25}$$

So,

$$I_2\left(t, B_x\left(t, x + \frac{p_k(t,x)}{\eta_k(t)}\right)\right) = I_2\left(t, \frac{\eta_k(t)V_x(t,x)}{\lambda(t)}\right)$$

Simplify the above equation we get

$$x + \frac{p_k(t,x)}{\eta_k(t)} = I_2\left(t, \frac{\eta_k(t)V_x(t,x)}{\lambda(t)}\right),$$

yielding

$$p_k^*(t,x) = \begin{cases} \max\left\{0, \left[I_2\left(t, \frac{\eta_k(t)V_x(t,x)}{\lambda(t)}\right) - x\right]\eta_k(t)\right\}, & if \ k = k^*(t) \\ 0, & Otherwise \end{cases}$$

Computing the second derivative with respect to each variable, we obtain

$$\begin{aligned} \mathcal{H}_{cc}(t,x;v^*) &= U_{cc}(t,c^*) < 0, \\ \mathcal{H}_{p_{k_1}p_{k_2}}(t,x;v^*) &= \frac{\lambda(t)}{\eta_{k_1}(t)\eta_{k_2}(t)} B_{xx}\left(t, x + \frac{p^*_{k^*}(t)}{\eta_{k^*(t)}(t)}\right) < 0, \\ \mathcal{H}_{\theta\theta}(t,x;v^*) &= x^2 V_{xx}(t,x)\sigma\sigma^T. \end{aligned}$$

It is enough to show that $\mathcal{H}_{\theta\theta}$ is negative definite. Note that $\sigma\sigma^T$ is assumed to be non-singular and so is σ, hence $\sigma\sigma^T$ is positive definite. In addition, $V_{xx}(t,x) < 0$: if $V_{xx}(t,x)$ was positive, then $\mathcal{H}$ would not be bounded above, and by the HJB equation, either $V_t(t,x)$ or $V(t,x)$ would have to be infinity, contradicting the smoothness assumption on V. Therefore $\mathcal{H}_{\theta\theta}$ is negative definite and so $\mathcal{H}$ has a unique maximum. □

2.5 The family of discounted CRRA utilities

Assume that the utility functions for the economic-agent are Constant Relative Risk Aversion (CRRA) and are given by

$$\begin{aligned} U(t,c) &= e^{-\rho t}\frac{c^\gamma}{\gamma}, \\ B(t,Z) &= e^{-\rho t}\frac{Z^\gamma}{\gamma}, \\ W(X) &= e^{-\rho T}\frac{X^\gamma}{\gamma}. \end{aligned} \tag{2.26}$$

where the risk parameter $\gamma < 1$, $\gamma \neq 0$ and the discount rate $\rho > 0$.

2.5.1 Optimal strategies

Using Theorem 2.4.1, we state the following optimal strategies for CRRA utility functions.

PROPOSITION 1. *[13] Assume Assumptions above are satisfied. Let ξ be a non-singular square matrix given by $(\sigma\sigma^T)^{-1}$. The optimal strategies in the case of CRRA utility functions are*

$$\begin{aligned} c^*(t,x) &= \frac{1}{o(t)}(x + b(t)), \\ \theta^*(t,x) &= \frac{1}{1-\gamma}\frac{x+b(t)}{x}\xi\alpha(t), \end{aligned}$$

$$p_k^*(t,x) = \begin{cases} \max\left\{0, \eta_k(t)\Big((D(t)-1)x + D(t)b(t)\Big)\right\}, & if\ k = k^*(t) \\ 0 & Otherwise \end{cases},$$

where

$$b(t) = \int_0^T i(s) e^{-\int_t^s \left(r(v)+\eta_{k^*(v)}(v) \right) dv} ds, \tag{2.27}$$

and

$$\begin{aligned}
D(t) &= \frac{1}{o(t)} \left(\frac{\lambda(t)}{\eta_{k^*(t)}} \right)^{\frac{1}{(1-\gamma)}}, \\
o(t) &= e^{-\int_t^T H(v)dv} + \int_t^T (e^{-\int_t^s H(v)dv}) L(s) ds, \\
H(t) &= \frac{\lambda(t)+\rho}{1-\gamma} - \frac{\gamma}{1-\gamma}\Big(r(t) + \eta_k(t) \Big) - \frac{\gamma}{(1-\gamma)^2}\Sigma(t), \\
L(t) &= 1 + \left(\frac{\lambda(t)}{(\eta_k(t))^{\gamma}} \right)^{\frac{1}{1-\gamma}}, \\
\Sigma(t) &= \alpha^T(t)\xi\alpha(t) - \frac{1}{2}\|\sigma^T\xi\alpha(t)\|^2.
\end{aligned}$$

Proof. Assume that U, B and W are as given in (2.26). Using the results in equation (2.21) and (2.22), we get

$$V_x(t,x) = U_x(t,c^*).$$

Substituting the value of U_x in the last identity yields

$$V_x(t,x) = e^{-\rho t}(c^*(t,x))^{\gamma-1}.$$

Then the optimal strategies of c and θ in terms of V are given by

$$c^*(t,x) = \Big(e^{\rho t} V_x(t,x) \Big)^{\frac{-1}{(1-\gamma)}},$$

$$\theta^*(t) = -\frac{V_x(t,x)}{xV_{xx}(t,x)}\xi\alpha(t).$$

Using equation (2.25) we get

$$B_x\left(t, x + \frac{p_k(t,x)}{\eta_k(t)}\right) = \frac{\eta_k(t)V_x(t,x)}{\lambda(t)}.$$

Using the definition of B_x, we get

$$e^{-\rho t}\left(x+\frac{p_k(t,x)}{\eta_k(t)}\right)^{\gamma-1}=\frac{\eta_k(t)V_x(t,x)}{\lambda(t)}.$$

Rearrange the last identity to arrive

$$p_k^*(t,x)=\begin{cases}\max\left\{0,\left(\left(\frac{\eta_k(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{-1}{(1-\gamma)}}-x\right)\eta_k(t)\right\}, & if\ k=k^*(t)\\ 0, & Otherwise\end{cases}.$$

Now we will solve the HJB equation (2.15). We will substitute the optimal strategies c, θ and p_k in the HJB equation but we are going to do that in 4 steps as below:

Step 1: For

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p_k(t)}{\eta_k(t)}\right)-V_x(t,x)p_k\right\}.$$

We substitute the value of B along with the optimal strategy p^* we get that

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p_k(t)}{\eta_k(t)}\right)-V_x(t,x)p_k\right\}$$

$$=\lambda(t)e^{-\rho t}\frac{\left[\cancel{x}+\frac{\left(\left(\frac{\eta_{k^*}(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-\cancel{x}\right)\cancel{\eta_{k^*}(t)}}{\cancel{\eta_{k^*}(t)}}\right]^{\gamma}}{\gamma}$$

$$-V_x(t,x)\left(\left(\frac{\eta_{k^*}(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-x\right)\eta_{k^*}(t).$$

Rearrange the terms in the equation above to obtain that

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p_k(t)}{\eta_k(t)}\right)-V_x(t,x)p_k\right\}$$

$$=\frac{\lambda(t)e^{-\rho t}\left(\eta_k(t)\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t\gamma}{\gamma-1}}\left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}}}{\gamma\left(\lambda(t)\right)^{\frac{\gamma}{\gamma-1}}}$$

$$+\eta_k(t)xV_x(t,x)-\frac{\left(\eta_k(t)\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t}{\gamma-1}}\left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}}}{\left(\lambda(t)\right)^{\frac{1}{\gamma-1}}},$$

that is

$$= \frac{\left(\lambda(t)\right)^{\frac{1}{1-\gamma}}}{\gamma} e^{\frac{\rho t}{\gamma-1}} \left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}} \left(\eta_k(t)\right)^{\frac{\gamma}{\gamma-1}}$$

$$-\left(\eta_k(t)\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{\rho t}{\gamma-1}} \left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}} \left(\lambda(t)\right)^{\frac{1}{1-\gamma}} + \eta_k(t) x V_x(t,x).$$

By taking out common factors we get:

$$\sup_{p\in\mathbb{R}_0^+} \left\{\lambda(t) B\left(t, x + \frac{p_k(t)}{\eta_k(t)}\right) - V_x(t,x) p_k\right\}$$

$$= \left(\frac{1-\gamma}{\gamma}\right) e^{\frac{\rho t}{\gamma-1}} \left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}} \left(\eta_k(t)\right)^{\frac{\gamma}{\gamma-1}} \left(\lambda(t)\right)^{\frac{1}{1-\gamma}} + \eta_k(t) x V_x(t,x).$$

Step 2:
Substitute the value of U and c we obtain

$$\sup_{c\in\mathbb{R}_0^+} \left\{U(t,c) - cV_x(t,x)\right\} = \frac{e^{-\rho t}(e^{\rho t} V_x(t,x))^{\frac{\gamma}{\gamma-1}}}{\gamma} - \left(e^{\rho t} V_x(t,x)\right)^{\frac{1}{\gamma-1}} V_x(t,x)$$

$$= e^{\frac{\rho t}{\gamma-1}} (\frac{1-\gamma}{\gamma}) \left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}}.$$

Step 3:
Now combine step 1 and step 2 and takeing out common factors we get.
$\sup_{c\in\mathbb{R}_0^+} \left\{U(t,c) - cV_x(t,x)\right\} + \sup_{p\in\mathbb{R}_0^+} \left\{\lambda(t) B\left(t, x + \frac{p_k(t)}{\eta_k(t)}\right) - V_x p_k\right\}$

$$= e^{\frac{\rho t}{\gamma-1}} \left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}} (\frac{1-\gamma}{\gamma}) \left(\frac{\left(\lambda(t)\right)^{\frac{1}{1-\gamma}}}{\left(\eta_k(t)\right)^{\frac{\gamma}{1-\gamma}}} + 1\right) + \eta_k(t) x V_x(t,x).$$

Step 4: For,

$$\sup_{\theta\in\mathbb{R}} \left\{\frac{x^2}{2} \sum_{m=1}^{M} \left(\sum_{n=1}^{N} \theta_n \sigma_{nm}(t)\right)^2 V_{xx}(t,x)\right\} = \frac{1}{2} \frac{V_x^2}{V_{xx}} \|\sigma^T \xi \alpha(t)\|^2,$$

and

$$\sup_{\theta\in\mathbb{R}} \left\{\sum_{n=1}^{N} \theta_n (\mu_n - r(t)) x V_x\right\} = \frac{-V_x^2}{V_{xx}} \alpha^T \xi \alpha.$$

Combining the above two we obtain

$$\sup_{\theta\in\mathbb{R}}\left\{\frac{x^2}{2}\sum_{m=1}^{M}(\sum_{n=1}^{N}\Theta_n\sigma_{nm}(t))^2V_{xx}(t,x)+\sum_{n=1}^{N}\Theta_n(\mu_n-r(t))xV_x\right\}=$$

$$\frac{-V_x^2}{V_{xx}}(\alpha^T\xi\alpha-\frac{1}{2}\|\sigma^T\xi\alpha(t)\|^2)=\frac{-V_x^2}{V_{xx}}\Sigma(t).$$

So the HJB equation will become:

$$V_t(t,x)-\lambda(t)V(t,x)+\left(i(t)+\left(r(t)+\eta_{k^*(t)}(t)\right)x\right)V_x(t,x)-\Sigma(t)\frac{(V_x(t,x))^2}{V_{xx}(t,x)}$$

$$+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{(1-\gamma)}}\left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}}L(t)=0, \tag{2.28}$$

where $\Sigma(t)$ and $L(t)$ are as given in Proposition 1.
We consider an *ansatz* of the form

$$V(t,x)=\frac{a(t)}{\gamma}(x+b(t))^{\gamma}.$$

Differentiate V with respect to t and x, we get

$$\begin{aligned}
V_t(t,x) &= a(t)(x+b(t))^{\gamma-1}\frac{db(t)}{dt}+(x+b(t))^{\gamma}\frac{da(t)}{dt}\frac{1}{\gamma},\\
V_x(t,x) &= a(t)(x+b(t))^{\gamma-1},\\
V_{xx}(t,x) &= (\gamma-1)a(t)(x+b(t))^{\gamma-2}.
\end{aligned}$$

Substitute the above partial derivatives in equation (2.28) and dividing by $(x+b(t))^{\gamma}$, then we get:

$$\frac{a(t)}{x+b(t)}\frac{db(t)}{dt}+\frac{da(t)}{dt}\frac{1}{\gamma}-\frac{\lambda(t)}{\gamma}a(t)+\left(i(t)+\left(r(t)+\eta_{k^*(t)}(t)\right)x\right)\frac{a(t)}{x+b(t)})$$

$$+\sum(t)\frac{a(t)}{1-\gamma}+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)(a(t))^{\frac{-\gamma}{1-\gamma}}=0.$$

Add and subtract the terms $\frac{r(t)b(t)a(t)}{x+b(t)}$ and $\frac{\eta(t)a(t)b(t)}{x+b(t)}$ to the L.H.S of the above equation we get,

$$\frac{1}{\gamma}\frac{da(t)}{dt}+\frac{a(t)}{x+b(t)}\frac{db(t)}{dt}-\frac{\lambda(t)}{\gamma}a(t)+\frac{i(t)a(t)}{x+b(t)}+\frac{r(t)xa(t)}{x+b(t)}+$$

$$\frac{r(t)a(t)b(t)}{x+b(t)}-\frac{r(t)a(t)b(t)}{x+b(t)}+\frac{\eta_{k^*(t)}xa(t)}{x+b(t)}+\frac{\eta_{k^*(t)}a(t)b(t)}{x+b(t)}$$

$$-\frac{\eta_{k^*(t)}a(t)b(t)}{x+b(t)}+\Sigma(t)\frac{a(t)}{1-\gamma}+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)\Big(a(t)\Big)^{\frac{-\gamma}{1-\gamma}}=0.$$

Then the partial differential equation above can be divided into two partial differential equations one for $a(t)$ and the other for $b(t)$ as follows:

$$\frac{1}{\gamma}\frac{da(t)}{dt}+\Big(r(t)+\eta_{k^*(t)}(t)-\frac{\lambda(t)}{\gamma}+\frac{\Sigma(t)}{1-\gamma}\Big)a(t)+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)\Big(a(t)\Big)^{\frac{-\gamma}{1-\gamma}}=0,$$

$$a(T)=e^{-\rho T}, \tag{2.29}$$

and

$$\frac{a(t)}{x+b(t)}\frac{db(t)}{dt}+\frac{i(t)a(t)}{x+b(t)}-\frac{r(t)a(t)b(t)}{x+b(t)}-\frac{\eta_{k^*(t)}a(t)b(t)}{x+b(t)}=0. \tag{2.30}$$

Multiply equation (2.30) by $\frac{x+b(t)}{a(t)}$

$$\frac{db(t)}{dt}-\Big(r(t)+\eta_{k^*(t)}(t)\Big)b(t)+i(t)=0,$$

$$b(T)=0. \tag{2.31}$$

To find a solution to the boundary value problem (2.29), we write $a(t)$ in the form

$$a(t)=e^{-\rho t}\Big(o(t)\Big)^{1-\gamma}.$$

Then, we obtain a new boundary value problem for the function $o(t)$ of the form

$$\frac{do(t)}{dt}-H(t)o(t)+L(t)=0,$$

$$o(T)=1, \tag{2.32}$$

where $L(t)$ and $H(t)$ are as given in Proposition 1. Since equation (2.32) is a linear first order ordinary differential equation then its solution is

$$o(t)=e^{-\int_t^T H(v)dv}+\int_t^T e^{-\int_t^s H(v)dv}L(s)ds.$$

The solution of (2.29) is given by

$$a(t)=e^{-\rho t}\Big(e^{-\int_t^T H(v)dv}+\int_t^T e^{-\int_t^s H(v)dv}L(s)ds\Big)^{1-\gamma}.$$

Similarly, we solve (2.31) to get

$$b(t) = \int_t^T i(s) e^{-\int_t^s (r(v)+\eta_{k^*(v)}(v))dv} ds.$$

Note that

$$V_x = a(t)(x + b(t))^{\gamma-1}.$$

Substitute the value of V_x in c^* we get

$$\begin{aligned}
c^*(t,x) &= (e^{\rho t} a(t)(x+b(t))^{\gamma-1})^{\frac{-1}{1-\gamma}}, \\
&= e^{\frac{-\rho t}{1-\gamma}} (a(t))^{\frac{-1}{1-\gamma}} (x+b(t)), \\
&= \cancel{e^{\frac{-\rho t}{1-\gamma}}} \left(\cancel{e^{-\rho t}} \Big(o(t) \Big)^{1-\gamma} \right)^{\frac{-1}{1-\gamma}} (x+b(t)), \\
&= (o(t))^{-1}(x+b(t)).
\end{aligned}$$

This leads to

$$c^*(t,x) = \frac{1}{o(t)}(x+b(t)).$$

Also, note that

$$V_{xx} = (\gamma-1)a(t)(x+b(t))^{\gamma-2}.$$

Substituting V_x, V_{xx} in (2.24) to obtain

$$\theta^*(t,x) = \frac{1}{1-\gamma} \frac{x+b(t)}{x} \xi \alpha(t).$$

Now according to $p_k^*(t,x)$ we have

$$p_k^*(t,x) = \begin{cases} \max\{0, \left(\left(\frac{\eta_k(t) e^{\rho t} V_x(t,x)}{\lambda(t)} \right)^{\frac{-1}{(1-\gamma)}} - x \right) \eta_k(t)\}, & if\ k = k^*(t) \\ 0, & Otherwise. \end{cases}$$

For the case $k = k^*(t)$,

$$\begin{aligned}
p_k^*(t,x) &= \left(\left(\frac{\eta_k(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{-1}{(1-\gamma)}} - x\right)\eta_k(t) \\
&= \left(\left(\frac{\eta_k(t)e^{\rho t}a(t)(x+b(t))^{\gamma-1}}{\lambda(t)}\right)^{\frac{-1}{1-\gamma}} - x\right)\eta_k(t) \\
&= \left(\left(\frac{\eta_k(t)e^{\rho t}a(t)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}(x+b(t)) - x\right)\eta_k(t) \\
&= \left(\frac{\left(\eta_k(t)e^{\rho t}\left(e^{-\rho t}(o(t))^{1-\gamma}\right)\right)^{\frac{1}{\gamma-1}}(x+b(t))}{\lambda(t)} - x\right)\eta_k(t) \\
&= \left(\frac{\left(\eta_k(t)\right)^{\frac{1}{\gamma-1}}\cancel{e^{\frac{\rho t}{\gamma-1}}}\cancel{e^{\frac{-\rho t}{\gamma-1}}}o^{-1}(t)(x+b(t))}{\left(\lambda(t)\right)^{\frac{1}{\gamma-1}}} - x\right)\eta_k(t).
\end{aligned}$$

But

$$\frac{\left(\eta_k(t)\right)^{\frac{1}{\gamma-1}}}{o(t)\left(\lambda(t)\right)^{\frac{1}{\gamma-1}}} = \frac{1}{o(t)}\left(\frac{\lambda(t)}{\eta_k(t)}\right)^{\frac{1}{1-\gamma}} = D(t).$$

So, $p_k^*(t,x) = \begin{cases} \max\left\{0, \eta_k(t)\Big((D(t)-1)x + D(t)b(t)\Big)\right\}, & if\ k = k^*(t) \\ 0 & Otherwise, \end{cases}$

where $b(t)$ and $D(t)$ are as given in the statement of Proposition 1. $\square$

Remark 2. *[13] From Proposition 1, we notice that the optimal strategy for life-insurance selection and purchase is either to buy no insurance from any company or else to buy an optimal(positive) amount of life-insurance from a single company the one with the smallest premium-payout ratio. The case of not buying at all any insurance is the case in which the economic-agent is with a sufficiently large wealth and is too close to retirement age T. To show this, note that as* $t \to T$ *, we have that* $o(t) \to 1$ *and* $b(t) \to 0$. *Then, provided that* $\lambda(t) < \eta_{k^*(t)}(t)$, *we obtain that* $D(t) < 1$. *Hence, for sufficiently large of* x, *then term* $(D(t)-1)x$ *will dominate the term* $D(t)b(t)$ *and the quantity* $\eta_{k^*(t)}(t)((D(t)-1)x + D(t)b(t))$ *becomes negative and so the optimal premium rate* p^* *according to Proposition 1 is 0.*

Chapter 3

Optimal strategies for an economic-agent in a continuous time model within social security system

3.1 Model setup

We are now going to introduce a new idea which is to make the economic-agent contributes in the social security system while he is participating in the life-insurance market. To proceed we start similarly by introducing the financial and life-insurance market model.

3.2 Industrial market models

In this section, we introduce a simplified version of the financial and life-insurance market models together with the social security system and wealth process.

3.2.1 Simplified financial and life insurance market models

Following the paper presented in chapter 2, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ given by the $\mathbb{P}$ -augmentation of the filtration generated by one- dimensional Brownian motion $W(\cdot)$, $\sigma(W(s) : s \leq t)$ for $t \geq 0$.

Let T be a fixed time in the future that represent the time of retirement. Assume an economic agent invests his savings in a financial market composed of one risk free-asset and one risky asset such that their respective prices $S_0(t)$ and $S_1(t)$, evolve according to the following stochastic differential equations:

$$\begin{aligned} dS_0(t) &= r(t)S_0(t)dt, \\ dS_1(t) &= \mu(t)S_1(t)dt + \sigma(t)S_1(t)dW(t), \end{aligned}$$

where $W(t)$ is the one-dimensional Brownian motion, $r(t)$ is the riskless interest rate, $\mu(t) \in \mathbb{R}$ is the risky-asset appreciation rates, $\sigma(t)$ is the risky-asset volatility, and $\mathcal{F}_t$ represents the information available to any given agent observing the financial market during the time interval $[0, t]$. The assumptions on the parameters of the model are the same as those introduced in chapter 2 . We are going to consider the same life-insurance market as introduced in chapter 2.

3.2.2 Social Security System model

In this section, we study how the economic-agent have to participate in the social security system in order to protect his family in the future. We assume that the economic-agent contributes in the social security system by paying an amount $q(t)$ to the government for $t \in [0, \min\{T, \tau\}]$.
Now if the economic-agent dies at time $\tau \leq T$ while he is participating in the social security system, then the government has to pay to his estate an amount

$$\frac{q(t)}{h(t)},$$

where $t \in [\tau, \tau^*]$, $\tau^* = \max\{\tau_i, i = 1, 2, ..., m\}$, τ_i represents the lifetime of the i^{th} member in his close family (the parents, the spouse, children below 18, disabled close member and divorced/widow daughter), m is the total number of his close family members.
Notice that

$$h(t) : [0, T] \to \mathbb{R}^+$$

is assumed to be continuous and deterministic positive function which is determined by the government and

$$q(t) : [0, T] \to \mathbb{R}_0^+$$

is a non-negative deterministic function.

The participation in the social system ends when the economic-agent dies or achieves retirement age, whichever happens first. Therefore the economic-agent's total legacy to his estate at time $\tau \le T$ is given by

$$\bar{Z}(\tau) = X(\tau) + \frac{p(\tau)}{\eta(\tau)} + \frac{q(\tau)}{h(\tau)}. \tag{3.1}$$

If the economic-agent lives that long $\tau \ge T$, then the social system pays all the amount of money that the economic-agent paid before.

All the assumptions and concepts in this chapter are the same as in chapter 2 except where it is mentioned.

3.2.3 Wealth process

Let θ_0 be the fraction of his wealth invested in money market S_0. and θ_1 be the fraction of his wealth invested in stock market S_1, such that $\theta_0 + \theta_1 = 1$. The wealth process $X(t)$, $t \in [0, T \wedge \tau]$, where $T \wedge \tau = \min\{T, \tau\}$ is then defined by:

$$X(t) = x_0 + \int_0^t \Big(i(s) - c(s) - p(s) - q(s)\Big) ds + \int_0^t \frac{\theta_0(s) X(s)}{S_0(s)} dS_0(s)$$
$$+ \int_0^t \frac{\theta_1(s) X(s)}{S_1(s)} dS_1(s).$$

And it can be written in this form

$$X(t) = x + \int_0^t \Big(i(s) - c(s) - p(s) - q(s) + \Big(\theta_0(s) r(s) + \theta_1(s)\mu(s)\Big) X(s)\Big) ds$$
$$+ \theta_1(s)\sigma(s) X(s) dW(s), \tag{3.2}$$

where x_0 is the economic-agent's initial wealth. Differentiate equation (3.2) with respect to t to get the differential form below

$$dX(t) = \Big(i(t) - c(t) - p(t) - q(t) + \Big(\theta_0(t) r(t) + \theta_1(t)\mu(t)\Big) X(t)\Big) dt$$
$$+ \theta_1(t)\sigma(t) X(t) dW(t). \tag{3.3}$$

3.3 Optimal control problem

In this section, we will state the optimal control problem for the economic-agent whose aim is to find the optimal strategies that maximize the expected utility obtained from: the consumption for all $t \leq min\{T, \tau\}$; the value of his fortune at retirement date T if he lives that long; and the value of his legacy in the event of premature death.

That is find a strategy $v = (c(\cdot), \theta(\cdot), p(\cdot))$ which maximize the expected utility

$$E_{0,x}\left[\int_0^{\tau\wedge T} U(s, c(s))ds + B(\tau, \bar{Z}(\tau))\mathbf{1}_{[0,T]}(\tau) + W(X(T))\mathbf{1}_{(T,\infty)}(\tau)\right],$$

where $\mathbf{1}_A$ is the indicator function of the set A. U is the utility function of the economic-agent's consumption at some instant of time $t \in [0, T]$, W is the utility function at retirement time T, and B is the utility function legacy at time $t \in [0, T]$. Note that same assumptions as, in chapter 2, hold for the utility functions U, B and W.

3.4 Stochastic optimal control problem

In this section, we follow the same procedure used in chapter 2 to transform the stochastic optimal control problem to one with a fixed planning horizon, after that we state the dynamic programming principle and derive the HJB equation.
Let $\mathcal{A}(t, x)$ be the set of admissible strategies $v = (c(\cdot), \theta(\cdot), p(\cdot))$ for the dynamic of the wealth process with boundary condition $X(t) = x$. For any $v \in \mathcal{A}(t, x)$ we define

$$\begin{aligned} J(t, x, v) &= E_{t,x}\left[\int_t^{\tau\wedge T} U(s, c(s))ds + B(\tau, \bar{Z}(\tau))\mathbf{1}_{[0,T]}(\tau)\right. \\ &+ \left. W(X_{t,x}^v(T))\mathbf{1}_{(T,\infty)}(\tau)|\tau > t, \mathcal{F}_t\right]. \end{aligned}$$

The next Lemma is the transformation of the above control problem into a one with a fixed planning horizon and its proof is exactly the same as Lemma 2.3.1 in chapter 2.

LEMMA 3.4.1. *Assume all Assumptions above are satisfied. If* τ *is independent of the filtration* $\mathbb{F}$, *then*

$$J(t,x;v) = E_{t,x}\Bigg[\int_t^T \Big(G^+(s,t)U(s,c(s)) + g^-(s,t)B(s,\bar{Z}(s))\Big)ds$$

$$+G^+(T,t)W(X(T))|\mathcal{F}_t\Bigg],$$

where $G^+(s,t)$ and $g^-(s,t)$ are exactly as given in chapter 2. The optimal control problem can be now introduced in dynamic programming form. That is, find a control $v \in \mathcal{A}(t,x)$ such that the function V satisfies

$$V(t,x) = \sup_{v\in\mathcal{A}(t,x)} J(t,x,v) = J(t,x,v^*). \tag{3.4}$$

LEMMA 3.4.2. *(DPP). Assume all Assumptions are satisfied, then* $V(t,x)$ *satisfies the recursive relation*

$$V(t,x) = \sup_{v\in\mathcal{A}(t,x)} E\Bigg[exp\Big(-\textstyle\int_t^s \lambda(u)du\Big)V(s,X^v_{t,x}(s))$$

$$+\textstyle\int_t^s \Big(G^+(u,t)U(u,c(u)) + g^-(u,t)B(u,\bar{Z}^v_{t,x}(u))\Big)du|\mathcal{F}_t\Bigg].$$

The proof follows closely as Lemma 2.3.2.

Before we start with the next theorem and its proof, let us introduce the following definition and lemma.

Definition 26. *[3] Suppose* $G:[0,T]\to\mathbb{R}$ *is continuously differentiable, with* $G(0)=G(T)=0$, *where* G *is a deterministic function and not a stochastic process, then*

$$\int_0^T GdW = -\int_0^T G^{'}Wdt.$$

LEMMA 3.4.3. *[3] Let* G *be a function that satisfies the previous conditions, then*

$$\mathbb{E}\Bigg[\int_0^T GdW\Bigg] = 0.$$

Proof. By using Definition 26, we conclude

$$\mathbb{E}\Bigg[\int_0^T GdW\Bigg] = -\int_0^T G^{'}\mathbb{E}[W(t)]dt = 0.$$

□

Theorem 3.4.4. *(HJB- Equation). Assume all assumptions are satisfied and* $V \in \ C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$. *Then* V *satisfies the HJB equation*

$$V_t(t,x) - \lambda(t)V(t,x) + \sup_{(c,\theta,p)\in\mathbb{R}_0^+\times\mathbb{R}\times\mathbb{R}_0^+} \mathcal{H}(t,x;c,\theta,p) = 0, \tag{3.5}$$

$$V(T,x) = W(x),$$

where the Hamiltonian $\mathcal{H}$ *is given by*

$$\mathcal{H}(t,x;v) = \Bigg(i(t) - c(t) - p(t) - q(t) + \Bigg(r(t) + \theta(\mu(t) - r(t))\Bigg)x\Bigg)V_x(t,x) + \frac{x^2}{2}\Bigg(\theta(t,x)\sigma(t)\Bigg)^2 V_{xx}(t,x) + U(t,c) + \lambda(t)B\Bigg(t, x + \frac{p(t)}{\eta(t)} + \frac{q(t)}{h(t)}\Bigg).$$

Moreover, $v^* = (c^*(\cdot),\theta^*(\cdot),p^*(\cdot)) \in \mathcal{A}(t,x)$ *whose wealth* X^* *is optimal if and if for* $s \in [t,T]$ *we have*

$$V_t(s,X^*(s)) - \lambda(s)V(s,X^*(s)) + \mathcal{H}(s,X^*(s);v^*) = 0. \tag{3.6}$$

Proof. To proof (3.5), let $s = t+h$ in the dynamic programming principle. Note that the corresponding wealth process X satisfies SDE (2.5). Then by $It\hat{o}'s$ formula given in Theorem (1.4.1), we have

$$\begin{aligned} V(t+h,X(t+h)) &= V(t,x) + \int_t^{t+h}\Bigg\{ V_t(s,X(s)) \\ &+ V_x(s,X(s))[r(s)X(s) - c(s) - p(s) - q(s) \\ &+ i(s) + \theta(s)(\mu(s) - r(s))X(s)] \\ &+ \frac{1}{2}V_{xx}(s,X(s))\theta^2(s)\sigma^2(s)X^2(s)\Bigg\}\, ds \\ &+ \int_t^{t+h} V_x(s,X(s))\theta(s)\sigma(s)X(s)dW(s). \end{aligned} \tag{3.7}$$

For h is small, and by using Taylor series expansion we notice that

$$\exp\Bigg\{ -\int_t^{t+h}\lambda(v)dv\Bigg\} = 1 - \lambda(t)h + O(h^2), \tag{3.8}$$

where $O(h^2)$ is some error of order two. According to equation (3.8) and the DPP in Lemma 3.4.2, we get that

$$\begin{aligned} 0 &= \sup_{(c,p,\theta)\in\mathcal{A}(t,x)} E\Bigg([1 - \lambda(t)h + O(h^2))V(t+h,X(t+h)) - V(t,x) \\ &+ \int_t^{t+h} g^-(u,t)B(u,Z(u)) + G^+(u,t)U(u,c(u))du \mid \mathcal{F}_t\Bigg]. \end{aligned}$$

Then insert (3.7) into the above equation, and divide the resulting equation by h, let $h \to 0$ and by using Lemma 3.4.3 we obtain

$$\begin{aligned} 0 &= \sup_{(c,p,\theta)} [V_t(t,x) - \lambda(t)V(t,x) + (r(t)x + \theta x(\mu(t) - r(t)) + i(t) - c(t) \\ &- p(t) - q(t))V_x(t,x) + \frac{1}{2}\sigma^2(t)\theta^2 x^2 V_{xx}(t,x) + \lambda(t)B(t,Z(t)) + U(t,c(t))]. \end{aligned}$$

Since $Z(t) = x + p(t)/\eta(t) + q(t)/h(t)$, and $V_t(t,x) - \lambda(t)V(t,x)$ does not depend on v, then equation (3.5) follows. □

Proof. Recalling that the expected value of the Itoˆ integral is equal to zero, we are now going to proof the second part of the HJB theorem given in (3.6) as follows.

Let $(c,p,\theta) \in \mathcal{A}(t,x)$ with the corresponding wealth X, applying ito's formula to $exp\{-\int_t^s \lambda(v)dv\}V(s,X(s))$ we get

$$\begin{aligned} V(t,x) &= e^{-\int_t^T \lambda(v)dv} W(X(T)) - \int_t^T e^{-\int_t^u \lambda(v)dv} \Big\{ V_t(u,X(u)) \\ &+ V_x(u,X(u))[i(u) - c(u) - p(u) - q(u) + r(u)X(u) + \theta(u)(\mu - r(u))] \\ &- \lambda(u)V(u,X(u)) + \frac{1}{2}V_{xx}(u,X(u))\theta^2(u)\sigma^2(u)X^2(u)\Big\}du \\ &- \int_t^T exp^{-\int_t^u \lambda(v)dv} V_x(u,X(u))\theta(u)\sigma(u)X(u)dW(u). \end{aligned}$$

Now, take the expectation of the above equation to obtain

$$\begin{aligned} V(t,x) &= \mathbb{E}\Big[e^{-\int_t^T \lambda(v)dv} W(X(T))\Big] - \mathbb{E}\Big[\int_t^T e^{-\int_t^u \lambda(v)dv} \Big\{ V_t(u,X(u)) \\ &- \lambda(u)V(u,X(u)) + V_x(u,X(u))[i(u) - c(u) - p(u) - q(u) \\ &+ r(u)X(u) + \theta(u)(\mu - r(u))] + \frac{1}{2}V_{xx}(u,X(u))\theta^2(u)\sigma^2(u)X^2(u)\Big\}du\Big] \\ &= J(t,x;c,p,\theta) - \mathbb{E}\Big[\int_t^T exp\Big\{-\int_t^u \lambda(v)dv\Big\}\Big\{V_t(u,X(u)) \\ &\quad - \lambda(u)V(u,X(u)) + \mathcal{H}(u,X(u);c,p,\theta)du\Big] \qquad (3.9) \\ &\geq J(t,x;c,p,\theta) - \mathbb{E}\Big[\int_t^T exp\Big\{-\int_t^u \lambda(v)dv\Big\}\Big\{V_t(u,X(u)) \\ &- \lambda(u)V(u,X(u)) + \sup_{(c,p,\theta)\in\mathcal{A}(t,x)} \mathcal{H}(u,X(u);c,p,\theta)du\Big] \qquad (3.10) \end{aligned}$$

$$= J(t,x;c,p,\theta).$$

So, from (3.9) we notice that

$$\begin{aligned} V(t,x) &= J(t,x;c^*,p^*,\theta^*) - \mathbb{E}\bigg[\int_t^T exp\Big\{-\int_t^u \lambda(v)dv\Big\}\Big\{V_t(u,X^*(u)) \\ &- \lambda(u)V(u,X^*(u)) + \mathcal{H}(u,X^*(u);c^*,p^*,\theta^*)du\bigg]. \end{aligned}$$

Since $V(t,x) - J(t,x;c^*,p^*,\theta^*) \geq 0$, we conclude that

$$V_t(s,X^*(s)) - \lambda(s)V(s,X^*(s)) + \mathcal{H}(s,X^*(s);v^*) \leq 0. \tag{3.11}$$

Also from (3.10), we get

$$\begin{aligned} V(t,x) &\geq J(t,x;c^*,p^*,\theta^*) - \mathbb{E}\bigg[\int_t^T exp\Big\{-\int_t^u \lambda(v)dv\Big\}\Big\{V_t(u,X^*(u)) \\ &- \lambda(u)V(u,X^*(u)) + \mathcal{H}(u,X^*(u);c^*,p^*,\theta^*)du\bigg]. \end{aligned}$$

Since

$$V(t,x) = J(t,x,c^*,p^*,\theta^*),$$

then

$$V_t(s,X^*(s)) - \lambda(s)V(s,X^*(s)) + \mathcal{H}(s,X^*(s);v^*) \geq 0. \tag{3.12}$$

Combining (3.11) and (3.12) gives (3.6).
The inverse part follow closely from (3.6). This completes the proof □

3.5 Implicit optimal strategies

In this section, we state the optimal strategies in terms of the value function and its derivatives.

Theorem 3.5.1. *Assume all assumptions are satisfied and that the value function* $V \in C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$. *Then* $\mathcal{H}$ *has a unique maximum* $v^* = (c^*(\cdot),\theta^*(\cdot),p^*(\cdot)) \in \mathcal{A}(t,x)$, *and the optimal strategies are given by*

$$c^*(t,x) = I_1\Big(t, V_x(t,x)\Big), \tag{3.13}$$

$$\theta^*(t,x) = -\frac{\alpha_0 V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}, \tag{3.14}$$

$$p^*(t,x) = \left(I_2\left(t,\eta(t)\Big(\lambda(t)\Big)^{-1}V_x(t,x)\right) - x - \frac{q(t)}{h(t)}\right)\eta(t), \tag{3.15}$$

where I_1 and I_2 are as given in (2.16) and (2.17) respectively and

$$\alpha_0 = \mu(t) - r(t).$$

Proof. By Theorem 3.4.4, we start by determining the maximum for $\mathcal{H}$ such that

$$\sup_{(c,\theta,p)\in\mathbb{R}_0^+\times\mathbb{R}\times\mathbb{R}_0^+} \mathcal{H}(t,x;c,\theta,p) = \sup_{c\in\mathbb{R}_0^+}\Big\{U(t,c) - cV_x(t,x)\Big\} + r(t)xV_x(t,x)$$

$$+\sup_{p\in\mathbb{R}_0^+}\Big\{\lambda(t)B\Big(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\Big) - V_x(t,x)p(t)\Big\} + (i(t)-q(t))V_x(t,x)$$

$$+\sup_{\theta\in\mathbb{R}}\Big\{\frac{x^2}{2}\Big(\theta(t)\sigma(t)\Big)^2\times V_{xx}(t,x) + \theta\Big(\mu(t)-r(t)\Big)xV_x(t,x)\Big\}. \tag{3.16}$$

Differentiate equation (3.16) with respect to c, θ , p respectively we get the following three conditions:

$$U_x(t,c^*) - V_x(t,x) = 0. \tag{3.17}$$

$$x^2V_{xx}(t,x)\theta\sigma^2 + (\mu(t)-r(t))xV_x(t,x) = 0, \tag{3.18}$$

$$\frac{\lambda(t)}{\eta(t)}B_x\Big(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\Big) - V_x(t,x) = 0. \tag{3.19}$$

By using the definition of the unique function I_1 and solving equation (3.17) for the control variable c, we get:

$$U_x(t,c^*) = V_x(t,x).$$

So,

$$I_1\Big(t,U_x(t,c^*)\Big) = I_1\Big(t,V_x(t,x)\Big) = c^*(t,x).$$

And now solve equation (3.18) for the control variable θ, we get

$$x^2V_{xx}(t,x)\theta^*(t,x)\sigma^2 = -\alpha_0 xV_x(t,x).$$

So,

$$\theta^*(t,x) = \frac{-\alpha_0 V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}.$$

Now solve equation (3.19) for the control variable p we obtain

$$\frac{\lambda(t)}{\eta(t)}B_x\Big(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\Big) = V_x(t,x),$$

$$B_x\left(t, x + \frac{p(t)}{\eta(t)} + \frac{q(t)}{h(t)}\right) = \frac{\eta(t)V_x(t,x)}{\lambda(t)},$$
$$I_2\left(t, B_x\left(t, x + \frac{p(t)}{\eta(t)} + \frac{q(t)}{h(t)}\right)\right) = I_2\left(t, \frac{\eta(t)V_x(t,x)}{\lambda(t)}\right).$$

Rearrange the above terms we obtain

$$p^*(t,x) = \left[I_2\left(t, \frac{\eta(t)V_x(t,x)}{\lambda(t)}\right) - x - \frac{q(t)}{h(t)}\right]\eta(t).$$

Finally, computing the second derivative with respect to each variable, we obtain

$$\mathcal{H}_{cc}(t,x;v^*) = U_{cc}(t,c^*),$$
$$\mathcal{H}_{pp}(t,x;v^*) = \frac{\lambda(t)}{\eta(t)\eta(t)} B_{xx}\left(t, x + \frac{p^*(t)}{\eta(t)} + \frac{q^*(t)}{h(t)}\right),$$
$$\mathcal{H}_{\theta\theta}(t,x;v^*) = x^2\sigma^2 v_{xx}(t,x).$$

Note that $\mathcal{H}_{cc}$ and $\mathcal{H}_{pp}$ are negative by the assumptions. Furthermore $\mathcal{H}_{\theta\theta}$ is negative by the same reason introduced in chapter 2. □

3.6 Family of discounted CRRA utilities

In this section, we will use the CRRA utility functions:

$$U(t,c) = e^{-\rho t}\frac{c^\gamma}{\gamma},$$

$$B(t,\bar{Z}) = e^{-\rho t}\frac{\bar{Z}^\gamma}{\gamma}, \tag{3.20}$$

and

$$W(X) = e^{-\rho T}\frac{X^\gamma}{\gamma},$$

where γ is the risk aversion parameter such that $\gamma < 1$, $\gamma \neq 0$, $\rho > 0$, and $\bar{Z}$ is as in (3.1).

3.6.1 Explicit optimal strategies

By using the optimal strategies obtained in Theorem 3.5.1 and applying the CRRA utilities, we get the following Proposition.

PROPOSITION 2. *Assume all assumptions are satisfied. Then the optimal strategies using CRRA utility functions are given by*

$$c^*(t,x) = \frac{1}{o(t)}(x + b(t)),$$
$$\theta^*(t,x) = \frac{\alpha_0(t)}{1-\gamma}.\frac{x + b(t)}{x\sigma^2(t)},$$
$$p^*(t,x) = \eta(t)\Bigg(\bigg(D(t) - 1\bigg)x + D(t)b(t) - \frac{q(t)}{h(t)}\Bigg),$$

where

$$\begin{aligned}
b(t) &= \int_t^T \left(i(s) - q(s) + \frac{\eta(s)q(s)}{h(s)}\right) e^{-\int_t^s \left(r(v)+\eta(v)\right)dv} ds,\\
D(t) &= \frac{1}{o(t)}\left(\frac{\lambda(t)}{\eta(t)}\right)^{\frac{1}{(1-\gamma)}},\\
o(t) &= e^{-\int_t^T H(v)dv} + \int_t^T \left(e^{-\int_t^s H(v)dv}\right)L(s)ds,\\
H(t) &= \frac{\lambda(t)+\rho}{1-\gamma} - \frac{\gamma}{1-\gamma}\bigg(r(t)+\eta(t)\bigg) + \frac{\gamma}{(1-\gamma)^2}\Sigma(t),\\
L(t) &= 1 + \left(\frac{\eta^\gamma(t)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}},\\
\Sigma(t) &= \frac{-\alpha_0^2(t)}{2\sigma^2},\\
\alpha_0(t) &= \mu(t) - r(t).
\end{aligned}$$

Proof. We start the proof by noticing that

$$U_x(t,c^*) = V_x(t,x).$$

Substitute the value of U_x, we get

$$e^{-\rho t}c^{\gamma-1} = V_x(t,x).$$

Rearrange the above equation for c we obtain

$$c^*(t,x) = \bigg(e^{\rho t}V_x(t,x)\bigg)^{\frac{1}{\gamma-1}}. \tag{3.21}$$

Substituting the value of B_x in equation (3.19), we obtain

$$\frac{\lambda(t)}{\eta(t)}e^{-\rho t}\left(x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)^{\gamma-1}=V_x(t,x). \tag{3.22}$$

Rearranging equation (3.22), we get

$$p^*(t,x)=\left(\left(\frac{V_x(t,x)e^{\rho t}\eta(t)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-x-\frac{q(t)}{h(t)}\right)\eta(t). \tag{3.23}$$

We remark that the optimal strategy θ^* will stay the same and is given by

$$\theta^*(t,x)=-\frac{\alpha_0 V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}. \tag{3.24}$$

We need to find a solution for the HJB equation by using the above optimal strategies (3.21), (3.23),(3.24). We are going to find the $\sup \mathcal{H}$ in four steps:
Step 1: Substitute the value of c^* from (3.21) and the value of $U(t,x)$ from (3.20), to get

$$\sup_{c\in\mathbb{R}_0^+}\left\{U(t,c)-cV_x(t,x)\right\}$$

$$=\frac{e^{-\rho t}\left(\left(e^{\rho t}V_x(t,x)\right)^{\frac{1}{\gamma-1}}\right)^{\gamma}}{\gamma}-\left(e^{\rho t}V_x(t,x)\right)^{\frac{1}{\gamma-1}}V_x(t,x).$$

Rearrange the above terms we obtain

$$\sup_{c\in\mathbb{R}_0^+}\left\{U(t,c)-cV_x(t,x)\right\} = \left(\frac{1-\gamma}{\gamma}\right)\left(V_x(t,x)\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t}{\gamma-1}}. \tag{3.25}$$

Step 2: Substitute the value θ from (3.24), so

$$\sup_{\theta\in\mathbb{R}}\left\{\tfrac{x^2}{2}\left(\theta(t)\sigma(t)\right)^2\times V_{xx}(t,x)+\theta(t)\alpha_0(t)xV_x(t,x)\right\}$$

$$=\frac{x^2}{2}\left(\frac{-\alpha_0(t)V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}\right)^2\sigma^2(t)\times V_{xx}(t,x)+\frac{-\alpha_0(t)V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}\alpha_0(t)xV_x(t,x).$$

Rearrange the above terms, we obtain

$$\sup_{\theta\in\mathbb{R}}\left\{\frac{x^2}{2}\left(\theta(t)\sigma(t)\right)^2\times V_{xx}(t,x)+\theta(t)\alpha_0(t)xV_x(t,x)\right\} = \frac{-1}{2}\frac{\alpha_0^2(t)V_x^2(t,x)}{V_{xx}(t,x)\sigma^2(t)}.$$

Step 3: Substitute the value of B from (3.20) to get

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)-V_x(t,x)p(t)\right\}$$

$$=\frac{\lambda(t)}{\gamma}e^{-\rho t}\left(x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)^{\gamma}-V_x(t,x)p(t).$$

Now substitute the value of p^* from (3.23) to obtain that

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)-V_x(t,x)p(t)\right\}$$

$$=\lambda(t)e^{-\rho t}\frac{\left[x+\frac{\left(\left(\frac{\eta(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-x-\frac{q(t)}{h(t)}\right)\cancel{\eta(t)}}{\cancel{\eta(t)}}+\frac{q(t)}{h(t)}\right]^{\gamma}}{\gamma}$$

$$-V_x(t,x)\left(\left(\frac{\eta(t)e^{\rho t}V_x(t,x)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-x-\frac{q(t)}{h(t)}\right)\eta(t).$$

Rearrange the first term above we get that

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)-V_x(t,x)p(t)\right\}$$

$$=\frac{e^{-\rho t}\lambda(t)}{\gamma}\left(\frac{V_x(t,x)e^{\rho t}\eta(t)}{\lambda(t)}\right)^{\frac{\gamma}{\gamma-1}}-V_x(t,x)\left(\left(\frac{V_x(t,x)e^{\rho t}\eta(t)}{\lambda(t)}\right)^{\frac{1}{\gamma-1}}-x-\frac{q(t)}{h(t)}\right)\eta(t)$$

$$=\frac{\lambda(t)}{\gamma}e^{\frac{\rho t}{\gamma-1}}\left(\frac{V_x(t,x)\eta(t)}{\lambda(t)}\right)^{\frac{\gamma}{\gamma-1}}-e^{\frac{\rho t}{\gamma-1}}\left(\frac{V_x(t,x)\eta(t)}{\lambda(t)^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma-1}}+\eta(t)\left(x+\frac{q(t)}{h(t)}\right)V_x(t,x).$$

So,

$$\sup_{p\in\mathbb{R}_0^+}\left\{\lambda(t)B\left(t,x+\frac{p(t)}{\eta(t)}+\frac{q(t)}{h(t)}\right)-V_x(t,x)p(t)\right\}=\eta(t)\left(x+\frac{q(t)}{h(t)}\right)V_x(t,x)$$

$$+\left(\frac{1-\gamma}{\gamma}\right)e^{\frac{\rho t}{\gamma-1}}\left(\frac{V_x(t,x)\eta(t)}{(\lambda(t))^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma-1}}.$$

Step 4: So, we get that

$$\sup\mathcal{H}(t,x;v)=\left(\frac{1-\gamma}{\gamma}\right)\Big(V_x(t,x)\Big)^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t}{\gamma-1}}+\Big(r(t)x+i(t)-q(t)\Big)V_x(t,x)$$

$$+\frac{-1}{2}\frac{\alpha_0^2(t)V_x^2(t,x)}{V_{xx}(t,x)\sigma^2(t)}+\frac{1-\gamma}{\gamma}\left(\frac{V_x(t,x)\eta(t)}{(\lambda(t))^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma-1}}e^{\frac{\rho t}{\gamma-1}}+V_x(t,x)\left(x+\frac{q(t)}{h(t)}\right)\eta(t).$$

Thus,

$$\sup\mathcal{H}(t,x;v)=e^{\frac{\rho t}{\gamma-1}}\Big(V_x(t,x)\Big)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}\right)\left[1+\frac{\Big(\eta(t)\Big)^{\frac{\gamma}{\gamma-1}}}{\Big(\lambda(t)\Big)^{\frac{1}{\gamma-1}}}\right]$$

$$+V_x(t,x)\left(\eta(t)\left(x+\frac{q(t)}{h(t)}\right)+r(t)x+i(t)-q(t)\right)+\frac{-1}{2}\frac{\alpha_0^2(t)V_x^2(t,x)}{V_{xx}(t,x)\sigma^2(t)}.$$

Substituting in the HJB equation (3.5), we get

$$V_t(t,x)-\lambda(t)V(t,x)+e^{\frac{\rho t}{\gamma-1}}\Big(V_x(t,x)\Big)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}\right)\left[1+\frac{\Big(\eta(t)\Big)^{\frac{\gamma}{\gamma-1}}}{\Big(\lambda(t)\Big)^{\frac{1}{\gamma-1}}}\right]$$

$$+V_x(t,x)\left(\eta(t)\left(x+\frac{q(t)}{h(t)}\right)+r(t)x+i(t)-q(t)\right)+\frac{-1}{2}\frac{\alpha_0^2(t)V_x^2(t,x)}{V_{xx}(t,x)\sigma^2(t)}=0.$$

For simplicity let

$$L(t)=1+\frac{\Big(\eta(t)\Big)^{\frac{\gamma}{\gamma-1}}}{\Big(\lambda(t)\Big)^{\frac{1}{\gamma-1}}},$$

$$\sum(t)=\frac{-\alpha_0^2}{2\sigma^2(t)}.$$

So the HJB equation (3.5) will become:

$$V_t(t,x)-\lambda(t)V(t,x)+e^{\frac{\rho t}{\gamma-1}}\Big(V_x(t,x)\Big)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}\right)L(t)$$

$$+\sum(t)\frac{V_x^2(t,x)}{V_{xx}(t,x)}+V_x(t,x)\left(\eta(t)\left(x+\frac{q(t)}{h(t)}\right)+r(t)x+i(t)-q(t)\right)=0, \quad (3.26)$$

and the terminal condition is given by:

$$V(T,x)=W(x). \quad (3.27)$$

Now let us consider the following ansatz function

$$V(t,x) = \frac{a(t)}{\gamma}\Big(x + b(t)\Big)^{\gamma}. \tag{3.28}$$

Differentiate $V(t,x)$ with respect to t and x as follows

$$\begin{aligned}
V_t(t,x) &= a(t)\Big(x+b(t)\Big)^{\gamma-1}.\frac{db(t)}{dt} + \frac{1}{\gamma}\Big(x+b(t)\Big)^{\gamma}.\frac{da(t)}{dt}, \\
V_x(t,x) &= a(t)\Big(x+b(t)\Big)^{\gamma-1}, \\
V_{xx}(t,x) &= (\gamma-1)a(t)\Big(x+b(t)\Big)^{\gamma-2}.
\end{aligned}$$

Substituting V, V_x, V_{xx} and V_t in the HJB equation (3.26) we get:

$$a(t)\Big(x+b(t)\Big)^{\gamma-1}\frac{db(t)}{dt} + \frac{1}{\gamma}\Big(x+b(t)\Big)^{\gamma}.\frac{da(t)}{dt} - \lambda(t)\frac{a(t)}{\gamma}\Big(x+b(t)\Big)^{\gamma}$$

$$\left(a(t)\Big(x+b(t)\Big)^{\gamma-1}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{1-\gamma}{\gamma}\right)L(t)e^{\frac{\rho t}{\gamma-1}} + \sum(t)\frac{\left(a(t)\Big(x+b(t)\Big)^{\gamma-1}\right)^2}{(\gamma-1)a(t)\Big(x+b(t)\Big)^{\gamma-2}}$$

$$+\left(x\Big(r(t)+\eta(t)\Big) + \frac{\eta(t)q(t)}{h(t)} + i(t) - q(t)\right)a(t)\Big(x+b(t)\Big)^{\gamma-1} = 0. \tag{3.29}$$

Dividing equation (3.29) by $(x+b(t))^{\gamma}$ and rearrange the terms we obtain

$$\begin{aligned}
&\frac{a(t)}{x+b(t)}\frac{db(t)}{dt} + \frac{da(t)}{dt}\frac{1}{\gamma} - \frac{\lambda(t)}{\gamma}a(t) \\
&+\left(x\Big(r(t)+\eta(t)\Big) + \frac{\eta(t)q(t)}{h(t)} + i(t) - q(t)\right)\frac{a(t)}{x+b(t)} \\
&+\sum(t)\frac{a(t)}{\gamma-1} + \frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)\Big(a(t)\Big)^{\frac{-\gamma}{1-\gamma}} = 0,
\end{aligned} \tag{3.30}$$

where $\sum(t)$ and $L(t)$ are as given in the statement of Proposition 2. Add and subtract the terms $\frac{r(t)b(t)a(t)}{x+b(t)}$ and $\frac{\eta(t)a(t)b(t)}{x+b(t)}$ from the L.H.S of equation (3.30), to get

$$\frac{1}{\gamma}\frac{da(t)}{dt} + \frac{a(t)}{x+b(t)}\frac{db(t)}{dt} - \frac{\lambda(t)}{\gamma}a(t) + \left(\frac{\eta(t)q(t)}{h(t)} + i(t) - q(t)\right)\frac{a(t)}{x+b(t)} +$$

$$\frac{r(t)xa(t)}{x+b(t)}+\frac{r(t)a(t)b(t)}{x+b(t)}-\frac{r(t)a(t)b(t)}{x+b(t)}+\frac{\eta(t)xa(t)}{x+b(t)}$$

$$+\frac{\eta(t)a(t)b(t)}{x+b(t)}-\frac{\eta(t)a(t)b(t)}{x+b(t)}+\Sigma(t)\frac{a(t)}{\gamma-1}+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)\Big(a(t)\Big)^{\frac{-\gamma}{1-\gamma}}=0.$$

The above differential equation can be divided into two independent boundary value problems for $a(t)$ and $b(t)$, respectively as follows

$$\frac{1}{\gamma}\frac{da(t)}{dt}-\frac{\lambda(t)}{\gamma}a(t)+\Sigma(t)\frac{a(t)}{\gamma-1}+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)(a(t))^{\frac{-\gamma}{1-\gamma}}$$
$$+\frac{\eta(t)a(t)\cancel{(x+b(t))}}{\cancel{(x+b(t))}}+\frac{r(t)a(t)\cancel{(x+b(t))}}{\cancel{(x+b(t))}}, \tag{3.31}$$

and

$$\frac{db(t)}{dt}.\frac{a(t)}{x+b(t)}+\left(\frac{\eta(t)q(t)}{h(t)}+i(t)-q(t)\right)\frac{a(t)}{x+b(t)}$$
$$-\frac{\eta(t)a(t)b(t)}{(x+b(t))}-\frac{r(t)a(t)b(t)}{(x+b(t))}=0. \tag{3.32}$$

Then equation (3.31) can be rewritten as

$$\frac{1}{\gamma}\frac{da(t)}{dt}+\left(r(t)+\eta(t)-\frac{\lambda(t)}{\gamma}+\frac{\sum(t)}{\gamma-1}\right)a(t)+\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)(a(t))^{\frac{-\gamma}{1-\gamma}}=0,$$
$$a(T)=e^{-\rho T}. \tag{3.33}$$

Multiplying equation (3.32) by $\frac{x+b(t)}{a(t)}$ then it can be rewritten as

$$\frac{db(t)}{dt}+\Big(-\eta(t)-r(t)\Big)b(t)+\left(\frac{\eta(t)q(t)}{h(t)}+i(t)-q(t)\right)=0,$$
$$b(T)=0. \tag{3.34}$$

To solve equation (3.33) we assume its solution has the form

$$a(t)=e^{-\rho t}(o(t))^{1-\gamma}. \tag{3.35}$$

Differentiate $a(t)$ with respect to time t, we get

$$\frac{da(t)}{dt}=e^{-\rho t}(1-\gamma)(o(t))^{-\gamma}\frac{do(t)}{dt}+(o(t))^{1-\gamma}(-\rho e^{-\rho t}).$$

Now substitute in equation (3.33) to obtain

$$\frac{1}{\gamma}\Bigg(e^{-\rho t}(1-\gamma)(o(t))^{-\gamma}\frac{do(t)}{dt}+(o(t))^{1-\gamma}.-\rho e^{-\rho t}\Bigg)+$$

$$\Bigg(r(t)+\eta(t)-\frac{\lambda(t)}{\gamma}+\frac{\sum(t)}{\gamma-1}\Bigg)e^{-\rho t}(o(t))^{1-\gamma}+$$

$$\frac{1-\gamma}{\gamma}e^{\frac{-\rho t}{1-\gamma}}L(t)\Bigg(e^{\rho t}\Bigg(o(t)\Bigg)^{1-\gamma}\Bigg)^{\frac{-\gamma}{1-\gamma}}=0.$$

Multiplying the above equation by $\frac{\gamma}{1-\gamma}$ and then take $e^{-\rho t}(o(t))^{-\gamma}$ as common factor, we get:

$$\frac{do(t)}{dt}+-\Bigg(\frac{\lambda(t)+\rho}{1-\gamma}-\frac{\gamma}{1-\gamma}(r(t)+\eta(t))+\frac{\gamma}{(1-\gamma)^2}\Sigma(t)\Bigg)o(t)+L(t)=0.$$

The above equation can be rewritten as the following:

$$\frac{do(t)}{dt}+-H(t)o(t)+L(t)=0, \tag{3.36}$$

$$o(T)=1, \tag{3.37}$$

where

$$H(t)=\frac{\lambda(t)+\rho}{1-\gamma}-\frac{\gamma}{1-\gamma}\Bigg(r(t)+\eta(t)\Bigg)+\frac{\gamma}{(1-\gamma)^2}\Sigma(t).$$

The above equation is a linear first order differential equation. so it can be solved explicitly by using the integrating factor method. The integrating factor is given by

$$\begin{aligned}\chi(t) &= e^{-\int_T^t H(u)du},\\ o(t) &= e^{\int_T^t H(u)du}\Bigg(\int_T^t -L(s)e^{-\int_T^s H(u)du}ds+C\Bigg).\end{aligned}$$

Applying the condition $e(T)=1$ gives $C=1$, so

$$o(t)=e^{-\int_t^T H(u)du}+\int_t^T L(s)e^{\int_s^T H(u)du}.e^{\int_T^t H(u)du}ds.$$

Rearranging $e(t)$ to get

$$o(t)=e^{-\int_t^T H(u)du}+\int_t^T L(s)e^{-\int_t^s H(u)du}ds. \tag{3.38}$$

Substituting the value of $o(t)$ in $a(t)$ given in (3.35) to get

$$a(t) = e^{-\rho t}\Bigg(e^{-\int_t^T H(u)du} + \int_t^T L(s)e^{-\int_t^s H(u)du}ds\Bigg)^{1-\gamma}. \tag{3.39}$$

Now we need to solve the second differential equation (3.34). It is again linear first order differential equation and its solution is given by

$$b(t) = e^{-\int_T^t(-\eta(u)-r(u))du}\Bigg(-\int_T^t\Bigg(\frac{\eta(s)q(s)}{h(s)} + i(s) - q(s)\Bigg)$$
$$.e^{\int_T^s(-\eta(u)-r(t))du} + C\Bigg).$$

As $b(T) = 0$, then $C = 0$, so we have

$$b(t) = \int_t^T\Bigg(\frac{\eta(s)q(s)}{h(s)} + i(s) - q(s)\Bigg).e^{-\int_t^s(\eta(u)+r(t))du}ds. \tag{3.40}$$

We already know that

$$c^*(t,x) = \Bigg(e^{\rho t}V_x(t,x)\Bigg)^{\frac{1}{\gamma-1}}.$$

Substituting the value of $V_x(t,x)$ in the above equation we get

$$c^*(t,x) = \Bigg(e^{\rho t}a(t)\Bigg(x+b(t)\Bigg)^{\gamma-1}\Bigg)^{\frac{1}{\gamma-1}}$$
$$= e^{\frac{\rho t}{\gamma-1}}\Bigg(a(t)\Bigg)^{\frac{1}{\gamma-1}}(x+b(t)).$$

Substituting the value of $a(t)$ given in (3.35), we get

$$c^*(t,x) = \frac{1}{o(t)}.(x+b(t)).$$

And we also know that

$$\theta^*(t,x) = \frac{-\alpha_0(t)V_x(t,x)}{xV_{xx}(t,x)\sigma^2(t)}.$$

Then substituting the value of $V_x(t,x)$ and $V_{xx}(t,x)$ in the last equation gives

$$\theta^*(t,x) = \frac{-\alpha_0(t)a(t)\Bigg(x+b(t)\Bigg)^{\gamma-1}}{x(\gamma-1)a(t)\Bigg(x+b(t)\Bigg)^{\gamma-2}\sigma^2(t)}.$$

Thus,

$$\theta^*(t,x) = \frac{\alpha_0(t)}{1-\gamma} \cdot \frac{x+b(t)}{x\sigma^2(t)}.$$

Substitute also the value of V_x in $p^*(t,x)$ given in equation (3.23), we get:

$$p^*(t,x) = \Bigg(\Bigg(\frac{V_x(t,x)e^{\rho t}\eta(t)}{\lambda(t)} \Bigg)^{\frac{1}{\gamma-1}} - x - \frac{q(t)}{h(t)} \Bigg)\eta(t),$$

$$= \Bigg(\Bigg(\frac{a(t)\Big(x+b(t)\Big)^{\gamma-1} e^{\rho t}\eta(t)}{\lambda(t)} \Bigg)^{\frac{1}{\gamma-1}} - x - \frac{q(t)}{h(t)} \Bigg)\eta(t),$$

$$= \Bigg(\frac{\cancel{e^{\frac{-\rho t}{\gamma-1}}}\Big((o(t))^{1-\gamma}\Big)^{\frac{1}{\gamma-1}}\Big(x+b(t)\Big)\cancel{e^{\frac{\rho t}{\gamma-1}}}\Big(\eta(t)\Big)^{\frac{1}{\gamma-1}}}{\Big(\lambda(t)\Big)^{\frac{1}{\gamma-1}}} - x - \frac{q(t)}{h(t)} \Bigg)\eta(t).$$

Thus,

$$p^*(t,x) = \Bigg(\frac{(o(t))^{-1}\Big(x+b(t)\Big)\Big(\eta(t)\Big)^{\frac{1}{\gamma-1}}}{\Big(\lambda(t)\Big)^{\frac{1}{\gamma-1}}} - x - \frac{q(t)}{h(t)} \Bigg)\eta(t).$$

For simplicity, we let

$$D(t) = \frac{1}{o(t)}\Bigg(\frac{\eta(t)}{\lambda(t)}\Bigg)^{\frac{1}{\gamma-1}}.$$

So $p^*(t,x)$ becomes

$$p^*(t,x) = \Bigg(D(t)\Big(x+b(t)\Big) - x - \frac{q(t)}{h(t)} \Bigg)\eta(t).$$

Finally,

$$p^*(t,x) = \Bigg(x\Big(D(t)-1\Big) + D(t)b(t) - \frac{q(t)}{h(t)} \Bigg)\eta(t).$$

This completes the proof. $\square$

Conclusion

We have introduced a continuous life-time model for an economic-agent who is investing his money in a financial market composed of one risk free asset and one risky asset while buying insurance to protect his family. Our model makes the economic-agent contributes in the social security system. We have determined the optimal strategies concerning consumption, investment and life-insurance within the social security system.

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